31st Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Day 1, June 7, 2000

1. The sequence $\langle a_n \rangle$ with $a_0 = 4$ and $a_1 = 1$ satisfies the recursion $a_{n+1} = a_n + 6a_{n-1}$ for $n \geq 1$ and defines the sequence
$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k.$$
Determine the coefficients α and β such that b_n satisfies the recursion
$$b_{n+1} = \alpha b_n + \beta b_{n-1}.$$
Furthermore, determine an explicit term for b_n.

2. The trapezoid $ABCD$ ($ABCD$ labled counterclockwise, $AB \parallel CD$) is inscribed into a circle k. On the arc $\widehat{AB}$ two points P and Q ($P \neq Q$) are chosen (with $APQB$ labled in counterclockwise order).

   Let X be the intersection of the lines CP and AQ and Y the of intersection of the lines BP and DQ.

   Show that P, Q, X and Y lie on a circle.

3. Determine all real solutions of the equation
$$\Big|\Big|\Big|\Big|\Big|\Big||x^2 - x - 1| - 3\Big| - 5\Big| - 7\Big| - 9\Big| - 11\Big| - 13\Big| = x^2 - 2x - 48.$$

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Day 2, June 8, 2000

4. In the acute-angled, non-isosceles triangle $\triangle ABC$ with angle $\gamma = 60°$ let U be the circumcenter, H the orthocenter and D the intersection of the lines AH and BC (that is, the orthogonal projection of A onto BC)

   Show that the Euler line HU is the bisector of the angle $\angle BHD$.

5. Determine all pairs of two integers (m, n) such that

$$\left|(m^2 + 2000m + 999999) - (3n^3 + 9n^2 + 27n)\right| = 1$$

   holds.

6. Determine all functions f mapping from the set of real numbers to the set of real numbers, such that for all real numbers x, y, z the relation $f(x + f(y + z)) + f(f(x + y) + z) = 2y$ holds.