

# 31st Austrian Mathematical Olympiad

## Federal Competition for Advanced Students

Day 1, June 7, 2000

1. The sequence  $\langle a_n \rangle$  with  $a_0 = 4$  and  $a_1 = 1$  satisfies the recursion  $a_{n+1} = a_n + 6a_{n-1}$  for  $n \geq 1$  and defines the sequence

$$b_n = \sum_{k=0}^n \binom{n}{k} a_k.$$

Determine the coefficients  $\alpha$  and  $\beta$  such that  $b_n$  satisfies the recursion

$$b_{n+1} = \alpha b_n + \beta b_{n-1}.$$

Furthermore, determine an explicit term for  $b_n$ .

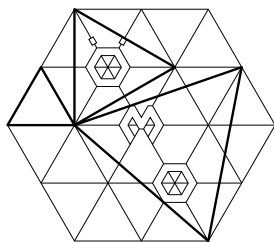
2. The trapezoid  $ABCD$  ( $ABCD$  labeled counterclockwise,  $AB \parallel CD$ ) is inscribed into a circle  $k$ . On the arc  $\widehat{AB}$  two points  $P$  and  $Q$  ( $P \neq Q$ ) are chosen (with  $APQB$  labeled in counterclockwise order).

Let  $X$  be the intersection of the lines  $CP$  and  $AQ$  and  $Y$  the of intersection of the lines  $BP$  and  $DQ$ .

Show that  $P$ ,  $Q$ ,  $X$  and  $Y$  lie on a circle.

3. Determine all real solutions of the equation

$$||| ||| |x^2 - x - 1| - 3| - 5| - 7| - 9| - 11| - 13| = x^2 - 2x - 48.$$



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4. In the acute-angled, non-isosceles triangle  $\triangle ABC$  with angle  $\gamma = 60^\circ$  let  $U$  be the circumcenter,  $H$  the orthocenter and  $D$  the intersection of the lines  $AH$  and  $BC$  (that is, the orthogonal projection of  $A$  onto  $BC$ )

Show that the Euler line  $HU$  is the bisector of the angle  $\angle BHD$ .

5. Determine all pairs of two integers  $(m, n)$  such that

$$|(m^2 + 2000m + 999999) - (3n^3 + 9n^2 + 27n)| = 1$$

holds.

6. Determine all functions  $f$  mapping from the set of real numbers to the set of real numbers, such that for all real numbers  $x, y, z$  the relation  $f(x + f(y + z)) + f(f(x + y) + z) = 2y$  holds.