

32nd Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Day 1, May 30, 2001

1. Prove that $\frac{1}{25} \sum_{k=0}^{2001} \left[\frac{2^k}{25} \right]$ is an integer.

(Here $[x]$ is the largest integer smaller or equal to x .)

2. Determine all triples of positive real numbers x , y and z such that both

$$x + y + z = 6 \quad \text{and}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2 - \frac{4}{xyz}$$

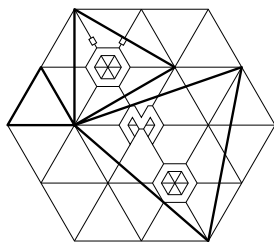
hold.

3. We are given a triangle $\triangle ABC$ and its circumcircle with midpoint U and radius r .

Let K be the circle with midpoint U and radius $2r$, and let c' be the tangent to K that is parallel to $c = AB$ and has the property that C lies between c and c' .

Analogously the tangents a' and b' are determined. The resulting triangle with sides a' , b' , c' is called $\triangle A'B'C'$.

Prove that the lines joining the midpoints of corresponding sides of the triangles ABC and $A'B'C'$ pass through a common point.



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4. Determine all real valued functions $f(x)$ in one variable such that the relation $f(f(x)^2 + f(y)) = xf(x) + y$ holds for all real numbers x and y .
5. Determine all integers m for which all solutions of the equation $3x^3 - 3x^2 + m = 0$ are rational.
6. We are given a semicircle over a line segment \overline{AB} . Points C and D are marked on the semicircle such that $\overline{AC} = \overline{CD}$. The tangent to the semicircle in C and the line joining B and D intersect in a point E . The line joining A and E intersects the semicircle in a point F .
Show that the inequality $\overline{CF} < \overline{FD}$ holds.