32nd Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Day 1, May 30, 2001

1. Prove that $\frac{1}{25}\sum_{k=0}^{2001}\left[\frac{2^k}{25}\right]$ is an integer.

   (Here $[x]$ is the largest integer smaller or equal to x.)

2. Determine all triples of positive real numbers x, y and z such that both

$$x+y+z=6 \quad \text{and}$$
$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=2-\frac{4}{xyz}$$

   hold.

3. We are given a triangle $\triangle ABC$ and its circumcircle with midpoint U and radius r.

   Let K be the circle with midpoint U and radius $2r$, and let c' be the tangent to K that is parallel to $c = AB$ and has the property that C lies between c and c'.

   Analogously the tangents a' and b' are determined. The resulting triangle with sides a', b', c' is called $\triangle A'B'C'$.

   Prove that the lines joining the midpoints of corresponding sides of the triangles ABC and $A'B'C'$ pass through a common point.

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Day 2, May 31, 2001

4. Determine all real valued functions $f(x)$ in one variable such that the relation $f(f(x)^2+f(y)) = xf(x)+y$ holds for all real numbers x and y.

5. Determine all integers m for which all solutions of the equation $3x^3-3x^2+m=0$ are rational.

6. We are given a semicircle over a line segment $\overline{AB}$. Points C and D are marked on the semicircle such that $\overline{AC}=\overline{CD}$. The tangent to the semicircle in C and the line joining B and D intersect in a point E. The line joining A and E intersects the semisicle in a point F.

   Show that teh inequality $\overline{CF} < \overline{FD}$ holds.