33rd Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 28, 2002

1. Determine all integers a and b such that $(19a+b)^{18}+(a+b)^{18}+(19b+a)^{18}$ is a square number.

2. Determine the largest real number C such that for all real numbers x and y $(x \neq y)$ with $xy = 2$ the inequality
$$\frac{((x+y)^2-6)((x-y)^2+8)}{(x-y)^2} \geq C$$
holds.

   For which pairs (x, y) does equality hold?

3. Let $f(x) = \frac{9^x}{9^x+3}$. Calculate the sum of all expressions of the form $f\left(\frac{k}{2002}\right)$ where the k is an integer between 0 and 2002 such that the fraction $\frac{k}{2002}$ is fully reduced.

4. We are given three mutually distinct points A, C and P in the plane. A and C are opposite corners of a parallelogram $ABCD$, the point P lies on the bisector of the angle DAB, and the angle APD is a right angle. Construct all possible parallelograms $ABCD$ that satisfy these conditions.

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Part 2, Day 1, June 5, 2002

1. On a regular 8×8 chess board there are many rectangles and squares composed only of whole squares on the board. Form the 1×1 square up to the 8×8 square.

   What is the sum of their areas?

   What is the formula in an "$a \times b$ chess board" for the sum of the areas of all rectangles and squares from the 1×1 square to the $a \times b$ rectangle?

2. Let $b > 800$ be an integer. Determine all $(a_1, a_2, \ldots, a_{2002})$ with natural numbers a_j $(1 \leq j \leq 2002)$ such that

$$\sum_{j=1}^{2002} a_j^{a_j} = 2002 b^b.$$

3. Let $ABCD$ and $AEFG$ be two similar cyclic quadrilaterals (labeled counterclockwise). Let P be the second point of intersection of the circumcircles of the quadrilaterals. Show that P lies on the line BE.

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Part 2, Day 2, June 6, 2002

4. Determine all polynomials $P(x)$ of the smallest possible degree with the following properties:

   (a) The coefficient of the highest power is 200.

   (b) The coefficient of the smallest appearing power is 2.

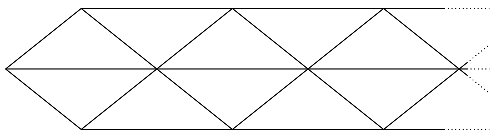
   (c) The sum of all coefficients is 4.

   (d) $P(-1) = 0$.

   (e) $P(2) = 6$.

   (f) $P(3) = 8$.

5. In a road network as drawn below, the vertices in the middle horizontal line are labled 1, 4, 7, ... The vertices in the upper row are labeled 2, 5, 8, ... and the vertices in the lower row 3, 6, 9, ...

   How many paths from "1" to "$3n+1$" exist such that vertices are visited only in increasing order.

6. We are given an acute-angled triangle ABC and its orthocenter H. HA, HB and HC divide the ABC into three small triangles ABH, BCH and CAH.

   Show that the three small triangles have the same circumference if and only if ABC is equilateral.