35th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 16, 2004

1. Determine all quadruples (a, b, c, d) of real numbers for which the following holds: The sum of the product of three numbers and the remaining number has the same value for each choice of the three numbers.

2. We are given a hexagon inscribed in a circle, and for its sides we have

$$\overline{AB} = \overline{BC} = a, \overline{CD} = \overline{DE} = b, \overline{EF} = \overline{FA} = c.$$

   Show that there are three (mutually disjoint) pairs of perpendicular diagonals.

3. For natural numbers a and b let $Z(a, b) = \frac{(3a)!(4b)!}{a!^4 b!^3}$.

   (a) Show that for $a \leq b$ the value $Z(a, b)$ is a natural number.

   (b) Show that for each natural number b there exists an infinite number of netural numbers a such that $Z(a, b)$ is not a natural number.

4. The $2N = 2004$ real numbers $x_1, x_2, \ldots, x_{2003}, x_{2004}$ are all either equal to $\sqrt{2} - 1$ or $\sqrt{2} + 1$.
   Can the sum $\sum_{k=1}^{N} x_{2k-1} x_{2k}$ have the value 2004?
   How many distinct integer values can it have? (With a suitable choice of the x_j.)

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Part 2, Day 1, April 26, 2004

1. Prove without the use of differential calculus:

   (a) For real numbers a, b, c, d, the inequality $a^6 + b^6 + c^6 + d^6 - 6abcd \geq -2$ holds. When does equality hold?

   (b) For which positive integers k does an equality of the form $a^k + b^k + c^k + d^k - kabcd \geq M_k$ hold? Determine the largest possible value for M_k and determine when equality holds.

2. (a) Show that for each set of preimes $\{p_1, p_2, \ldots, p_k\}$ the following holds true: The sum of all unit fractions (i.e. fractions of the form $\frac{1}{n}$) with denominators consisting of exactly the k given prime numbers (in arbitrary powers with non-zero exponent) is again a unit fraction.

   (b) What is the value of this sum if $\frac{1}{2004}$ is one of the summands?

   (c) Show that for each sets $\{p_1, p_2, \ldots, p_k\}$ of k primes ($k > 2$) the described sum is smaller than $\frac{1}{N}$ with $N = 2 \cdot 3^{k-2}(k-2)!$.

3. In a trapezoid $ABCD$ with circumcircle k the diagonals AC and BD are perpendicular. Two circles k_a and k_c are drawn whose diameters are AB and CD respectively.

   Calculate the circumference and the area of the region that lies within the circumcircle k, but outside of the circles k_a and k_c.

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Part 2, Day 2, May 27, 2004

4. Show that there exists an infinite sequence $\langle a_1, a_2, \ldots, a_n, \ldots \rangle$ of positive integers such that for all N the sum $\sum_{k=1}^{N} {a_k}^2$ is a square number.

   Give a recursion formula for such a sequence.

5. Solve the following system of equations over real numbers.

$$a^2 = \frac{\sqrt{bc}\sqrt[3]{bcd}}{(b+c)(b+c+d)}$$
$$b^2 = \frac{\sqrt{cd}\sqrt[3]{cda}}{(c+d)(c+d+a)}$$
$$c^2 = \frac{\sqrt{da}\sqrt[3]{dab}}{(d+a)(d+a+b)}$$
$$d^2 = \frac{\sqrt{ab}\sqrt[3]{abc}}{(a+b)(a+b+c)}$$

6. Triangles ABP, BCQ and CAR are drown outside of an equilateral triangle ABC with area 1, where the angles at P, Q and R are equal to $60°$.

   (a) What is the largest area the triangle PQR can have?

   (b) What is the largest area the triangle of the incenters of ABP, BCQ and CAR can have?