

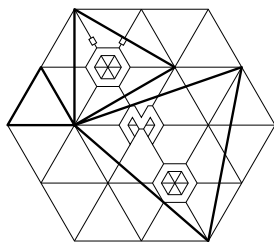
36th Austrian Mathematical Olympiad
Federal Competition for Advanced Students
Part 1, May 30, 2005

1. Show that there exist infinitely many multiples of 2005 that contain all 10 digits $0, 1, 2, 3, \dots, 9$ equally often (without counting leading zeros).
2. For how many integer values a with $|a| \leq 2005$ does the system of equations

$$\begin{aligned}x^2 &= y + a \\y^2 &= x + a\end{aligned}$$

have integer solutions?

3. For three real numbers a, b, c let $s_n = a^n + b^n + c^n$ be the sum of their n -th powers. It is known that $s_1 = 2$, $s_2 = 6$ and $s_3 = 14$. Show that for all integers $n > 1$ the identity $|s_n^2 - s_{n-1}s_{n+1}| = 8$ holds.
4. We are given two congruent equilateral triangles ABC and PQR with parallel sides. The one is "pointing up", the other one "pointing down". The intersection of the triangle areas is a hexagon. Show that the three diagonals of the hexagon that connect opposite vertices pass through one common point.



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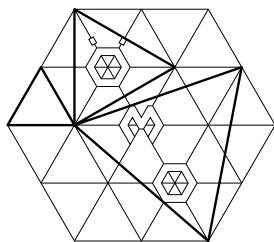
Federal Competition for Advanced Students

Part 2, Day 1, June 8, 2005

1. Determine all triples (a, b, c) of positive integers such that $a + b + c$ is the least common multiple of these three numbers.
2. Let a, b, c, d be positive real numbers.
Prove the inequality

$$\frac{a + b + c + d}{abcd} \leq \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3}.$$

3. In an acute-angled triangle ABC two circles k_1 and k_2 are drawn whose diameters are the sides AC and BC . Let E be the foot of the altitude h_b on AC and let F be the foot of the altitude h_a on BC .
Let L and N be the intersections of the line BE with the circle k_1 (L on the line BE) and let K and M be the intersections of the line AF with the circle k_2 (K on the line AF).
Show that $KLMN$ is a cyclic quadrilateral.



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Part 2, Day 2, June 9, 2005

4. The function f is defined on the set $\{0, 1, 2, \dots, 2005\}$ assuming non-negative integer values, and satisfies the following three conditions for all non-negative integers x for which the arguments lie in the given set:

$$f(2x + 1) = f(2x), f(3x + 1) = f(3x), f(5x + 1) = f(5x)$$

How many different values can be assumed by the function?

5. Determine all sextuples (a, b, c, d, e, f) of real numbers that satisfy the following system of equations:

$$4a = (b + c + d + e)^4$$

$$4b = (c + d + e + f)^4$$

$$4c = (d + e + f + a)^4$$

$$4d = (e + f + a + b)^4$$

$$4e = (f + a + b + c)^4$$

$$4f = (a + b + c + d)^4$$

6. Let Q be a point inside a cube. Show that there are infinitely many lines g passing through Q such that Q is the midpoint of the line segment PR of g that lies inside the cube.