36th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 30, 2005

1. Show that there exist infinitely many multiples of 2005 that contain all 10 digits $0, 1, 2, 3, \ldots, 9$ equally often (without counting leading zeros).

2. For how many integer values a with $|a| \leq 2005$ does the system of equations

$$x^2 = y + a$$
$$y^2 = x + a$$

   have integer solutions?

3. For three real numbers a, b, c let $s_n = a^n + b^n + c^n$ be the sum of their n-th powers.
   It is known that $s_1 = 2$, $s_2 = 6$ and $s_3 = 14$.
   Show that for all integers $n > 1$ the identity $|{s_n}^2 - s_{n-1}s_{n+1}| = 8$ holds.

4. We are given two congruent equilateral triangles ABC and PQR with parallel sides. The one is ”‘pointing up”’, the other one ”‘pointing down”’. The intersection of the triangle areas is a hexagon.
   Show that the three diagonals of the hexagon that connect opposite vertices pass through one common point.

36th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 2, Day 1, June 8, 2005

1. Determine all triples (a, b, c) of positve integers such that $a+b+c$ is the least common multiple of these three numbers.

2. Let a, b, c, d be positive real numbers.
   Prove the inequality
   $$\frac{a+b+c+d}{abcd} \leq \frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3}.$$

3. In an acute-angled triangle ABC two cicles k_1 and k_2 are drawn whose diameters are the sides AC and BC. Let E be the foot of the altitude h_b on AC and let F be the foot of the altitude h_a on BC.
   Let L and N be the intersections of the line BE with the circle k_1 (L on the line BE) and let K and M be the intersections of the line AF with the circle k_2 (K on the line AF).
   Show that $KLMN$ is a cyclic quadrilateral.

36th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 2, Day 2, June 9, 2005

4. The function f is defined on the set $\{0, 1, 2, \ldots, 2005\}$ assuming non-negative integer values, and satisfies the following three conditions for all non-negative integers x for which the arguments lie in the given set:

$$f(2x+1) = f(2x), f(3x+1) = f(3x), f(5x+1) = f(5x)$$

   How many different values can be assumed by the function?

5. Determine all sextuples (a, b, c, d, e, f) of real numbers that satisfy the following system of equations:

$$\begin{aligned} 4a &= (b+c+d+e)^4 \\ 4b &= (c+d+e+f)^4 \\ 4c &= (d+e+f+a)^4 \\ 4d &= (e+f+a+b)^4 \\ 4e &= (f+a+b+c)^4 \\ 4f &= (a+b+c+d)^4 \end{aligned}$$

6. Let Q be a point inside a cube.
   Show that there are infinitely many lines g passing through Q such that Q is the midpoint of the line segment PR of g that lies inside the cube.