

38th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 18, 2007

1. In a table with 2007 rows and 2007 columns and odd integer is written in each cell.

For $1 \leq i \leq 2007$ let Z_i be the sum of the numbers in the i -th row and for $1 \leq j \leq 2007$ let S_j be the sum of the numbers in the j -th column.

Further let A be the product of all Z_i and B the product of all S_j .

Show that $A + B$ is different from zero.

2. For each positive integer n determine the largest possible value $C(n)$ such that for all n -tuples (a_1, a_2, \dots, a_n) of mutually different integers the inequality

$$(n+1) \sum_{j=1}^n a_j^2 - \left(\sum_{j=1}^n a_j \right)^2 \geq C(n)$$

holds.

3. Let $M(n) = \{-1, -2, \dots, -n\}$. For each nonempty subset we calculate the product of its elements. What is the sum of all these products?
4. Let $n > 4$ be an integer.

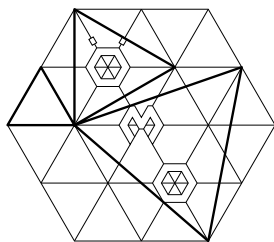
We are given a convex n -gon $A_0A_1A_2\dots A_{n-1}A_n$ (with $A_n = A_0$) that is inscribed into a circle, and whose side lengths are given as $A_{i-1}A_i = i$ ($1 \leq i \leq n$).

Furthermore let φ_i be the angle between the line A_iA_{i+1} and the tangent to the circle in the point A_i . (Note: The angle between two lines is always smaller or equal to 90° .)

Determine the sum

$$\Phi = \sum_{i=0}^{n-1} \varphi_i$$

of these n angles.



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Part 2, Day 1, June 5, 2007

1. For which non-negative integers $a < 2007$ does the congruence $x^2 + a \equiv 0 \pmod{2007}$ have exactly two different non-negative solutions smaller than 2007?
 (I.e., there exist exactly two non-negative integers u and v smaller than 2007 such that $u^2 + a$ and $v^2 + a$ are divisible by 2007.)
2. Determine all sextuples (x_1, x_2, \dots, x_6) of non-negative integers that satisfy the following system of equations.

$$x_1x_2(1 - x_3) = x_4x_5$$

$$x_2x_3(1 - x_4) = x_5x_6$$

$$x_3x_4(1 - x_5) = x_6x_1$$

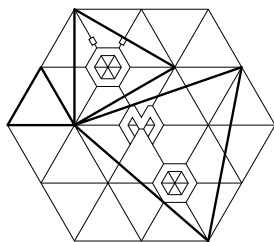
$$x_4x_5(1 - x_6) = x_1x_2$$

$$x_5x_6(1 - x_1) = x_2x_3$$

$$x_6x_1(1 - x_2) = x_3x_4$$

3. Determine all rhombi (diamonds) $ABCD$ with sides of length $2a$ by giving their angle $\alpha = \angle BAD$, such that the following holds:

There exists a circle that intersects each side of the rhombus in a line segment of length a .



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Part 2, Day 2, June 6, 2007

4. Let M be the set of all polynomials $P(x)$ whose roots are pairwise distinct integers, and which have the property that the absolute values of all coefficients smaller than 2007.

What is the highest degree among the polynomials in M ?

5. We are given a convex n -gon with a triangulation, that is, a division into triangles by non-intersecting diagonals.

Show that the n vertices can be marked with the digits of 2007 in such a way that each quadrilateral consisting of 2 neighbouring (along an edge) triangles has 9 as the sum of the digits at its 4 vertices.

6. We are given a triangle ABC with circumcircle k has midpoint U and radius r . On the extension of the radius UA a point P is chosen. The line PB is reflected about the line BA . The reflection is called g .

Likewise, the line PC is reflected about CA . The reflection is called h .

The intersection of g and h is called Q .

Determine the set of all points Q when P varies on the entire extension of UA beyond A (that is, the points on the ray UA outside of the circle k).