39th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 22, 2008

1. What is the smallest positive residue of
$$1 \cdot \binom{2008}{0} + 2 \cdot \binom{2008}{1} + 3 \cdot \binom{2008}{2} + \ldots + 2009 \cdot \binom{2008}{2008}$$
when divided by 2008?

2. Let a be a positive real number and $n > 4$ an integer.

   Determine all n-tupels $(x_1, x_2, \ldots, x_n)$ of positive real numbers that satisfy the following system of equations.
$$\begin{aligned} x_1x_2(3a - 2x_3) &= a^3 \\ x_2x_3(3a - 2x_4) &= a^3 \\ &\vdots \\ x_{n-2}x_{n-1}(3a - 2x_n) &= a^3 \\ x_{n-1}x_n(3a - 2x_1) &= a^3 \\ x_nx_1(3a - 2x_2) &= a^3 \end{aligned}$$

3. Let $p > 1$ be a natural number.

   We consider the set $\mathbf{F}_p$ of all non-constant sequences of non-negative integers that satisfy the recursion $a_{n+1} = (p + 1)a_n - pa_{n-1}$ (for all $n > 0$).

   Show that there exists a sequence $\langle a_n \rangle$ in $\mathbf{F}_p$ with the property that for every other sequence $\langle b_n \rangle$ in $\mathbf{F}_p$, the inequality $a_n \leq b_n$ holds for all n.

4. In a triangle ABC let E be the midpoint of the sides AC and F the midpoint of the side BC. Furthermore let G be the foot of the altitude through C on the side AB (or its extension).

   Show that the triangle EFG is isosceles if and only if ABC is isosceles.

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Part 2, Day 1, June 5, 2008

1. Show that for positive real numbers a, b, c with $a+b+c=1$ the inequality

$$\sqrt{a^{1-a}b^{1-b}c^{1-c}} \leq \frac{1}{3}$$

holds.

2. (a) Does there exist a polynomial $P(x)$ with integer coefficients such that for each positive integer d that divides 2008, one has $P(d) = \frac{2008}{d}$?

   (b) For which positive integers n do exist polynomials $P(x)$ with integer coefficients such that for each positive integers d that divides n, one has $P(d) = \frac{n}{d}$?

3. The line g is given, and on it the four points P, Q, R, S (in this order from left to right).

   Construct all squares $ABCD$ with the following properties:

   P lies on the line trough A and D.

   Q lies on the line trough B and C.

   R lies on the line trough A and B.

   S lies on the line trough C and D.

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Part 2, Day 2, June 6, 2008

4. Determine all functions f mapping the set of positive integers to the set of non-negative integers satisfying the following conditions:

   (a) $f(n \cdot m) = f(m) + f(n)$;

   (b) $f(2008) = 0$;

   (c) For all $n \equiv 39 \pmod{2008}$ one has $f(n) = 0$.

5. Which positive integers are missing in the sequence $\langle a_n \rangle$?

$$\left\langle a_n = n + \lfloor \sqrt{n} \rfloor + \lfloor \sqrt[3]{n} \rfloor \right\rangle_{n \geq 1}.$$

6. We are given a square $ABCD$. Let P be different from the vertices of the square and from its center M.

   For a point P for which the line PD intersects the line AC, let E be this intersection.

   For a point P for which the line PC intersects the line DB, let F be this intersection.

   All those points P for which E and F exist are called acceptable points.

   Determine the set of acceptable points for which the line EF is parallel to AD.