40th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 17, 2009

1. Show, that the following inequality holds for all positive integers n:

$$3^{n^2} > (n!)^4$$

2. We generalize the functions factorial $n! = n(n-1)(n-2)\cdots 2\cdot 1$ and double factorial $n!! = n(n-2)(n-4)\cdots 1$ for odd n and $n!! = n(n-2)(n-4)\cdots 2$ for even n. For $n > 0$, we define the k-multifactorial $F_k(n) = n(n-k)(n-2k)\cdots r$ with $1 \le r \le k$ and $n \equiv r \pmod{k}$. Further, $F_k(0) = 1$.

   Determine all non-negative integers n, such that $F_{20}(n) + 2009$ is a square number (the square of an integer).

3. Given is a circuit with ***n* stations** ($n > 1$) which can be traveled on in both directions. Each segment between two neighboring stations is called section. One of the stations is called Raach [the town where the ÖMO is held, ed.].

   A bus should start in Raach and, after passing through ***n* + 2 sections**, return to Raach. In doing so it should visit **each station at least once**.

   For all $n > 1$, determine the number $f(n)$ of tours which satisfy these conditions.

4. Let D, E and F be the midpoints of the sides of the triangle ABC (D on BC, E on CA and F on AB).

   Further let $H_aH_bH_c$ be the triangle formed by the base points of the altitudes of the triangle ABC.

   Let P, Q and R be the midpoints of the sides of the triangle $H_aH_bH_c$ (P on H_bH_c, Q on H_cH_a and R on H_aH_b).

   Show: The lines PD, QE and RF share a common point.

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Part 2, Day 1, June 10, 2009

1. $A_{km}(x)$ is an exponential tower made up of k twos at the bottom, one x and, finally, m more twos on top. $B_k(y)$ is also an exponential tower, made up of k fours at the bottom and one y on top. In other words,

$$A_{km}(x) = 2^{2^{\cdot^{\cdot^{\cdot^{2^{x^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}}}}}}} \qquad \text{and} \qquad B_k(y) = 4^{4^{\cdot^{\cdot^{\cdot^{4^{y}}}}}}.$$

   Determine all pairs (x, y) of non-negative integers, dependent on $k > 0$, such that $A_{kk}(x) = B_k(y)$.

   Comment: An exponential tower a^{b^c} is calculated as $a^{(b^c)}$.

2. (a) For positive integers $a < b$ let

$$M(a,b) = \frac{\sum_{k=a}^{b} \sqrt{k^2+3k+3}}{b-a+1}$$

   denote the arithmetic mean of the expressions $\sqrt{k^2+3k+3}$ for $a \le k \le b$. Calculate $K(a,b) = [M(a,b)]$.

   (b) Calculate

$$N(a,b) = \frac{\sum_{k=a}^{b} \left[\sqrt{k^2+3k+3}\right]}{b-a+1},$$

   the arithmetic mean of the expressions $\left[\sqrt{k^2+3k+3}\right]$ for $a \le k \le b$.

   (We assume that $[x]$ is the largest integer less than or equal to x.)

3. We are given a triangle ABC. Determine all points P in the interior of the triangle, such that the following holds:

   Let D be the common point of the continuation of AP with BC and let A' be the point on this continuation, for which $\overline{AD} = \overline{DA'}$ holds. The triangles ABC and $A'BC$ are congruent. (The points need not correspond in this order.) If B' and C' are defined in an analogous manner, triangles $AB'C$ and ABC' are also congruent to ABC.

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Part 2, Day 2, June 11, 2009

4. For each positive integer a we consider the sequence $\langle a_n \rangle$ with $a_0 = a$ and $a_n = a_{n-1} + 40^{n!}$ for $n > 0$.

   Prove that every such sequence contains infinitely many numbers that are divisible by 2009.

5. Let $n > 1$ and for $1 \leq k \leq n$ let $p_k = p_k(a_1, a_2, \ldots, a_n)$ be the sum of the products of all possible combinations of k of the numbers $a_1, a_2, \ldots, a_n$.

   Furthermore, let $P = P(a_1, a_2, \ldots, a_n)$ be the sum of all p_k with odd values of k less than or equal to n.

   How many different values are taken by the a_j, if all a_j $(1 \leq j \leq n)$ and P are prime?

6. Let $ABCD$ be a quadrilateral and let P, Q, R and S be the mid-points of the sides AB, BC, CD and DA respectively. We call the quadrilateral $PQRS$ *mid-point quadrilateral* of $ABCD$.

   Determine all circumscribed quadrilaterals whose mid-point quadrilaterals are squares.