43rd Austrian Mathematical Olympiad

Federal Competition for Advanced Students

Part 1, May 17, 2012

1. Determine all functions $f\colon\ :\mathbb{Z}\to\mathbb{Z}$ satisfying the following property:

   For each pair of numbers m and n (not necessarily distinct), $\mathrm{ggT}(m,n)$ divides $f(m)+f(n)$.

2. Determine all solutions of the equation $n!+A\cdot n=n^k$ with $n,k\in\mathbb{N}$ for $A=7$ and for $A=2012$.

3. Consider a strip of n fields, numbered from left to right with the integers 1 to n in ascending order. Each of the fields should be colored with one of the colors 1, 2 or 3. Even-numbered fields can be colored with any color. Odd-numbered fields are only allowed to be colored with the odd colors 1 and 3.

   How many such colorings are there such that any two neighboring fields have different colors?

4. Let ABC be a scalene (i.e. non-isosceles) triangle. Let U be the center of the circumcircle of this triangle and I the center of the incircle. Assume that the intersection of the angle bisector of $\gamma=\angle ACB$ with the circumcircle of ABC lies on the bisector of UI.

   Show that γ is the second-largest angle in the triangle ABC.

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Part 2, Day 1, June 6, 2012

1. Determine a number m as large as possible such that the following is true:

   For all non-zero real numbers a, b and c with $\left|\frac{1}{a}\right| + \left|\frac{1}{b}\right| + \left|\frac{1}{c}\right| \leq 3$ the inequality

   $$\left(a^2 + 4\left(b^2 + c^2\right)\right)\left(b^2 + 4\left(c^2 + a^2\right)\right)\left(c^2 + 4\left(a^2 + b^2\right)\right) \geq m$$

   holds. When does equality hold?

2. Solve the equation

   $$x^4y^3(y - x) = x^3y^4 - 216$$

   over the integers.

3. An isosceles trapezoid $PQRS$ is called *interesting*, if it is inscribed in a unit square $ABCD$ so that on each side of the square there is exactly one vertex of the trapezoid and such that the connecting lines between midpoints of (neighboring) sides of the trapezoid are parallel to the sides of the square.

   Determine all interesting trapezoids and their areas.

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Part 2, Day 2, June 7, 2012

4. Given is a sequence $\langle a_1,\, a_2,\, a_3,\, \ldots\rangle$ of real numbers. For each positive integer n let m_n be the arithmetic mean of the numbers a_1 to a_n.

   Suppose that there exists a real number C such that

   $$(i-j)m_k + (j-k)m_i + (k-i)m_j = C$$

   holds for all triples $(i,\, j,\, k)$ of distinct positive integers. Show that the sequence $\langle a_1,\, a_2,\, a_3,\, \ldots\rangle$ is an arithmetic sequence.

5. Determine the number of non-negative integers $N < 1\,000\,000 = 10^6$ with the following property:

   There is a integer k with $1 \leq k \leq 43$ such that 2012 is a divisor of $N^k - 1$.

6. Given is an equilateral triangle ABC with side length 2. We consider all equilateral triangles PQR with side length 1, for which the following holds:

   - P is on the side AB,
   - Q is on the side AC, and
   - R is inside or on the edge of the triangle ABC.

   Determine the set of all points inside the triangle ABC which are the barycenter of such a triangle PQR.

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