44th Austrian Mathematical Olympiad

Federal Competition for Advanced Students

<u>Part 1</u>, May 1, 2013

1. Show that if for non-negative integers m, n, N, k the equation

$$\left(n^2+1\right)^{2^k} \cdot \left(44n^3+11n^2+10n+2\right) = N^m$$

holds then $m = 1$.

2. Solve the following system of equations over the rational numbers:

$$\begin{aligned}(x^2+1)^3 &= y+1\\ (y^2+1)^3 &= z+1\\ (z^2+1)^3 &= x+1\end{aligned}$$

3. Put the positive integers into two lines as follows:

1		3			6					11								19		32		53		···
	2		4	5		7	8	9	10		12	13	14	15	16	17	18		20 to 31		33 to 52		54	···

First we write 1 in the upper line, then 2 in the lower line and 3 again in the upper line. Afterwards the further integers in the upper line are written individually, while in the lower line they are written in blocks. The number of (consecutive) integers in a block is determined by the the first number in the previous block.

Let a_1, a_2, a_3, … be the numbers in the upper line.

Give an explicit formula for a_n.

4. Let A, B and C be three points on a line (in this order).

For each circle k through the points B and C let D be one point of intersection of the bisector of B and C with the circle k. Further let E be the second point of intersection of the line AD with k.

Show that for each circle k the ratio of lengths $\overline{BE} : \overline{CE}$ is the same.

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Part 2, Day 1, May 29, 2013

1. Let $\lfloor x \rfloor$ be the largest integer smaller or equal to x and let $\lceil x \rceil$ be the smallest integer larger or equal to x.

   Determine for each pair (a, b) of positive integres all non-negative integers n with

   $$b + \left\lfloor \frac{n}{a} \right\rfloor = \left\lceil \frac{n+b}{a} \right\rceil .$$

2. Let k be an integer.

   Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and

   $$f(x^k y^k) = xyf(x)f(y) \qquad \text{for } x,\ y \neq 0.$$

3. Inscribed in a circle are a square and an equilateral triangle. The seven vertices form a convex septagon S inscribed in the circle (S might be a hexagon if one vertex of the triangle coincides with a vertex of the square).

   For with positions of the triangle relative to the square does S have the largest area and smallest area respectively?

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Part 2, Day 2, May 30, 2013

4. Let a_1, a_2, …, a_n be non-negative numbers, such that for all real numbers $x_1 > x_2 > \cdots > x_n > 0$ with $x_1 + x_2 + \cdots + x_n < 1$ the inequality $\sum_{k=1}^{n} a_k x_k^3 < 1$ holds.

   Show that
$$na_1 + (n-1)a_2 + \cdots + (n-j+1)a_j + \cdots + a_n \leq \frac{n^2(n+1)^2}{4}.$$

5. Let $n \geq 3$ be an integer. Let $A_1A_2 \ldots A_n$ be a convex n-gon. Consider a line g through A_1 which does not contain a further point of the circumference of the n-gon. Let h be the perpendicular to g through A_1. Project the n-gon orthogonally on h.

   For $j = 1, \ldots, n$, let B_j be the image of A_j. The line g is called admissible if the points B_j are distinct.

   Consider all convex n-gons and all admissible lines g. How many different orders of the points B_1, …, B_n are possible?

6. Consider a regular octahedron $ABCDEF$ with lower vertex E, upper vertex F, middle cross-section $ABCD$, midpoint M and circumscribed sphere k. Further let X be an arbitrary point inside of the facet ABF. Let the line EX intersect k in E and Z, and the plane $ABCD$ in Y.

   Show that $\sphericalangle EMZ = \sphericalangle EYF$.