31st Austrian Mathematical Olympiad

Regional Competition for Advanced Students

April 13, 2000

1. For which non-negative integers n does $2^n > 10n^2 - 60n + 80$ hold?

2. For each real number a determine all real numbers x such that the following equation holds.

$$(2x+1)^4 + ax(x+1) - \frac{a}{2} = 0$$

3. We consider two circles k_1 and k_2 with midpoints M_1 and M_2 and radii r_1 and r_2, with $z = M_1M_2 > r_1 + r_2$ and an outer tangent that touches the circles in P_1 and P_2 (both lying on the same side of the line M_1M_2). We now change the radii such that $r_1 + r_2 = c$ remains constant. Determine all possible locations for the midpoint of P_1P_2 when r_1 varies from 0 to c.

4. We consider the sequence $\langle u_n \rangle$ defined by the recursion $u_{n+1} = \dfrac{u_n(u_n+1)}{n}$ for $n \geq 1$.

   (a) Determine the elements of the sequence for $u_1 = 1$.

   (b) Show that if one element of the sequence is a non-integer rational number, then also all further sequence elements are not integers.

   (c) Show that for each non-negative integer K there exists a $u_1 > 1$ such that the first K elements of the sequence are integers.