33rd Austrian Mathematical Olympiad

Regional Competition for Advanced Students

April 23, 2002

1. Determine the smallest positive integer x such that all of the following fractions are completely reduced, i.e., their numerator and denominator are relatively prime.

$$\frac{3x+9}{8}, \frac{3x+10}{9}, \frac{3x+11}{10}, \ldots, \frac{3x+49}{48}$$

2. Solve the following system of equations over real numbers.

$$\begin{aligned} 2x_1 &= x_5^2 - 23 \\ 4x_2 &= x_1^2 + 7 \\ 6x_3 &= x_2^2 + 14 \\ 8x_4 &= x_3^2 + 23 \\ 10x_5 &= x_4^2 + 34 \end{aligned}$$

3. In a convex (all interior angles smaller than $180°$) hexagon $ABCDEF$ with circumference s the triangles ACE and BDF have circumference u and v, respectively.

   (a) Prove the inequalities $\frac{1}{2} < \frac{s}{u+v} < 1$.

   (b) Determine whether 1 can be replaced by a smaller or $\frac{1}{2}$ by a larger number such that the inequalities still hold for all convex hexagons.

4. Let $a_0, a_1, \ldots, a_{2002}$ be real numbers.

   (a) Show that the smallest of the values $a_k\,(1 - a_{2002-k})$ $(0 \le k \le 2002)$ is less than or equal to $\frac{1}{4}$.

   (b) Does this proposition also always hold true for the smallest of the values $a_k\,(1 - a_{2003-k})$ $(1 \le k \le 2002)$?

   (c) Show for positive real numbers $a_1, \ldots, a_{2002}$: The smallest of the values $a_k\,(1 - a_{2003-k})$ $(1 \le k \le 2002)$ is less than or equal to $\frac{1}{4}$.