35th Austrian Mathematical Olympiad

Regional Competition for Advanced Students

April 20, 2004

1. Determine all integers a and b such that $(a^3 + b)(a + b^3) = (a + b)^4$.

2. Solve the equation

$$\sqrt{4 - x\sqrt{4 - (x-2)\sqrt{1 + (x-5)(x-7)}}} = \frac{5x - 6 - x^2}{2}$$

   over the real numbers.
   (All square roots are non-negative.)

3. We are given a convex quadrilateral $ABCD$ with $\measuredangle ADC = \measuredangle BCD > 90^\circ$.
   Let E be the intersection of the line AC with the line parallel to AD through B and F be the intersection of the line BD with the line parallel to BC through A.
   Show that EF is parallel to CD.

4. The sequence $\langle x_n \rangle$ is defined by

$$x_{n+1} = \left(\frac{n}{2004} + \frac{1}{n}\right) {x_n}^2 - \frac{n^3}{2004} + 1 \text{ for } n > 0.$$

   Let x_1 be a positive integer smaller than 204 such that all elements of the sequence are positive integers.
   Show that the sequence then contains infinitely many primes.