# 37th Austrian Mathematical Olympiad

## Regional Competition for Advanced Students

April 27, 2006

1. Let $0 < x < y$ be real numbers, and let

$$H = \frac{2xy}{x+y}, \quad G = \sqrt{xy}, \quad A = \frac{x+y}{2}, \quad Q = \sqrt{\frac{x^2+y^2}{2}}$$

be the harmonic, geometric, arithmetic and quadratic means of $x$ and $y$. It is known that $H < G < A < Q$ holds.
Sort the intervals $[H, G]$, $[G, A]$ and $[A, Q]$ ascendingly by their lengths.

2. Let $n > 1$ be a natural number and $a$ a real number.
Determine all real solutions $(x_1, x_2, \ldots, x_n)$ of the following system of equations:

$$\begin{aligned} x_1 + a x_2 &= 0 \\ x_2 + a^2 x_3 &= 0 \\ &\vdots \\ x_k + a^k x_{k+1} &= 0 \\ &\vdots \\ x_{n-1} + a^{n-1} x_n &= 0 \\ a^n x_1 + x_n &= 0 \end{aligned}$$

3. In a non-isosceles triangle $ABC$ let $w$ be the bisector of the exterior angle in $C$ (exterior bisector of $\gamma$). The intersection of $w$ with the extension of $AB$ is $D$. Let now $k_A$ be the circumcircle of the triangle $ADC$ and analogously $k_B$ the circumcircle of the triangle $BDC$. Let $t_A$ be the tangent to $k_A$ in $A$ and analogously $t_B$ the tangent to $k_B$ in $B$. Let $P$ be the intersection of these two tangents.
Let now the points $A$ and $B$ be given.
Determine the set of the points $P = P(C)$ for all points $C$ such that $ABC$ is a non-isosceles, acute-angled triangle.

4. Let $\langle h_n \rangle_{n \in \mathbb{N}}$ be a harmonic sequence of positive rational numbers (i.e., each $h_n$ is the harmonic mean of its neighbors $h_{n-1}$ and $h_{n+1}$, that is, $h_n = \frac{2h_{n-1}h_{n+1}}{h_{n-1}+h_{n+1}}$).
Show that if the sequence contains an element $h_j$ that is the square of a rational number, then it contains infinitely many elements $h_k$ that are squares of rational numbers.