38th Austrian Mathematical Olympiad

Regional Competition for Advanced Students

April 24, 2007

1. Let $0 < x_0, x_1, \ldots, x_{669} < 1$ be mutually different real numbers.
   Show that there exists a pair x_i, x_j such that
   $$0 < x_i x_j (x_j - x_i) < \frac{1}{2007}$$

2. Determine all quintuples of positive integers $x_1 > x_2 > x_3 > x_4 > x_5 > 0$ with
   $$\left\lfloor \frac{x_1 + x_2}{3} \right\rfloor^2 + \left\lfloor \frac{x_2 + x_3}{3} \right\rfloor^2 + \left\lfloor \frac{x_3 + x_4}{3} \right\rfloor^2 + \left\lfloor \frac{x_4 + x_5}{3} \right\rfloor^2 = 38.$$
   (Remark: $\lfloor x \rfloor$ denotes the largest interger smaller than or equal to x.)

3. Let a be a poitive real number and n a non-negative integer.
   Compare the values of S and T:
   $$S = \sum_{k=-2n}^{2n+1} \frac{(k-1)^2}{a^{\left|\left\lfloor \frac{k}{2} \right\rfloor\right|}} \qquad T = \sum_{k=-2n}^{2n+1} \frac{k^2}{a^{\left|\left\lfloor \frac{k}{2} \right\rfloor\right|}}$$
   (Remark: $\lfloor x \rfloor$ denotes the largest interger smaller than or equal to x.)

4. In a convex quadrilateral $ABCD$ (all interior angles smaller than 180°) let M be the intersection of the diagonals.
   Determine all quadrilaterals for which there exists a line g through M that intersects the line segment AB in P and the line segment CD in Q such that the four triangles APM, BPM, CQM and DQM are similar. (The corners do not necessarily need to correspond in the given order.)