39th Austrian Mathematical Olympiad

Regional Competition for Advanced Students

April 24, 2008

1. Prove that for all real numbers a, b, c with $0 < a, b, c < 1$ the inequality

$$\sqrt{a^2bc + ab^2c + abc^2} + \sqrt{(1-a)^2(1-b)(1-c) + (1-a)(1-b)^2(1-c) + (1-a)(1-b)(1-c)^2} < \sqrt{3}.$$

holds.

2. For any real number x let $[x]$ be the next-smaller integer to x, i.e., the integer g with $g \leq x < g+1$ and let $\{x\} = x - [x]$ be the "decimal part of x".
Determine all triples of real numbers (a, b, c) satisfying the following system of equations:

$$\begin{aligned} \{a\} + [b] + \{c\} &= 2.9 \\ \{b\} + [c] + \{a\} &= 5.3 \\ \{c\} + [a] + \{b\} &= 4.0 \end{aligned}$$

3. We are given an acute-angled triangle ABC.
Determine all points P inside the triangle such that

$$1 \leq \frac{\angle APB}{\angle ACB}, \frac{\angle BPC}{\angle BAC}, \frac{\angle CPA}{\angle CBA} \leq 2$$

4. For each positive integer n let

$$a_n = \sum_{k=n}^{2n} \frac{(2k+1)^n}{k}$$

Show that a_n is not a natural number for any n.