40th Austrian Mathematical Olympiad

Regional Competition for Advanced Students

April 23, 2009

1. State a domain $M \subseteq \mathbb{R}^+$ as large as possible such that for all $a, b, c, d \in M$ the inequality

$$\sqrt{ab} + \sqrt{cd} \geq \sqrt{a+b} + \sqrt{c+d}$$

holds. Does

$$\sqrt{ab} + \sqrt{cd} \geq \sqrt{a+c} + \sqrt{b+d}$$

also hold for all $a, b, c, d \in M$? (Remark: $\mathbb{R}^+$ is the set of all positive real numbers.)

2. How many integer solutions $(x_0, x_1, x_2, x_3, x_4, x_5, x_6)$ does the equation

$$2x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 9$$

have?

3. Given is an acute-angled triangle ABC (counterclockwise labeled) with altitude base points D (on BC), E (on AC) and F (on AB). Further let P, Q and R be defined as follows:

   - P is the base point of the altitude of F on BC in the triangle CFB.
   - P is the base point of the altitude of D on AC in the triangle ADC.
   - P is the base point of the altitude of E on AB in the triangle AEB.

   The six points D, E, F, P, Q and R form with suitable numbering $T_1T_2T_3T_4T_5T_6$ (counterclockwise with $T_1 = P$) a convex hexagon (all angles smaller than 180 degrees).
   Show: In this convex hexagon there is no point which lies on all three diagonals T_1T_4, T_3T_6 and T_5T_2.

4. Two distinct arithmetic sequences $\langle a_0, a_1, \ldots, a_n = a_0 + nd, \ldots \rangle$ are essentially different, if they do not differ only be the absence of finitely many members at the beginning of one of them.
   How many pairwise essentially different non-constant arithmetic sequences of positive integers are there which contain an infinite non-constant geometric sequence whose third member is $40 \cdot 2009 = 80360$?