41st Austrian Mathematical Olympiad

Regional Competition for Advanced Students

April 15, 2010

1. Let $0 \leq a, b \leq 1$ be real numbers. Show that

$$\sqrt{a^3b^3} + \sqrt{(1-a^2)(1-ab)(1-b^2)} \leq 1.$$

2. Solve the equation

$$4x^4 - x^2(4y^4 + 4z^4 - 1) - 2xyz + y^8 + 2y^4z^4 + y^2z^2 + z^8 = 0$$

   for real x, y, z.

3. Let $\triangle ABC$ be a triangle with D a point on the side BC. Let U and V denote the centers of the circumcircles of the triangles $\triangle ABD$ and $\triangle ADC$ respectively. Show that the triangles $\triangle ABC$ and $\triangle AUV$ are similar.

4. Let $(b_n)_{n\geq 0} = \sum_{k=0}^{n}(a_0 + kd)$ for positive integers a_0 and d. Consider all sequences of this kind for which $b_i = 2010$ for some positive integer i. Determine the largest possible value of i and the corresponding values of a_0 and d.