42nd Austrian Mathematical Olympiad

Regional Competition for Advanced Students

April 12, 2011

1. Show that if $p_1, \ldots, p_{42}$ are 42 distinct prime numbers, then the sum $\sum_{j=1}^{42} \frac{1}{p_j^2+1}$ cannot be a unit fraction $\frac{1}{n^2}$ of a square number.

2. Determine all triples (x, y, z) of real numbers that satisfy the system of equations
$$2^{\sqrt[3]{x^2}} \cdot 4^{\sqrt[3]{y^2}} \cdot 16^{\sqrt[3]{z^2}} = 128,$$
$$\left(xy^2 + z^4\right)^2 = 4 + \left(xy^2 - z^4\right)^2 .$$

3. Given is a circle k with center M and a tangent t in the point T. A point $P \neq T$ is chosen on t. A line $g \neq t$ is drawn through P, intersecting the circle in two points $U \neq V$. Let S be the midpoint of the arc UV which does not contain the point T. Let Q be the point obtained by reflection of P on the line TS.

   Show that the four points Q, T, U and V are the vertices of a trapezoid.

4. We define the sequence (a_n) of positive integers by $a_1 = 1$ and such that a_{n+1} is the smallest positive integer such that for the smallest common multiples
$$\gcd(a_1, \ldots, a_{n+1}) > \gcd(a_1, \ldots, a_n)$$
   holds. Which integers are contained it the sequence?