44th Austrian Mathematical Olympiad

Regional Competition for Advanced Students

April 9, 2013

1. For which integers between 2000 and 2010 (including) is the probability that a random divisor is smaller or equal 45 the largest?

   (Hint: The probability is the number of divisors smaller or equal 45 divided by the total number of divisors.)

2. Determine all integers x satisfying

$$\left[\frac{x}{2}\right]\left[\frac{x}{3}\right]\left[\frac{x}{4}\right] = x^2.$$

   (Hint: $[y]$ is the largest integer which is not larger than y.)

3. For non-negative real numbers a, b let $A(a, b)$ be their arithmetic mean and $G(a, b)$ their geometric mean. We consider the sequence $\langle a_n \rangle$ with $a_0 = 0$, $a_1 = 1$ and $a_{n+1} = A(A(a_{n-1}, a_n), G(a_{n-1}, a_n))$ for $n > 0$.

   (a) Show that each $a_n = b_n^2$ is the square of a rational number (with $b_n \geq 0$).

   (b) Show that the inequality $\left|b_n - \frac{2}{3}\right| < \frac{1}{2^n}$ holds for all $n > 0$.

4. We call a pentagon distinguished if either all side lengths or all angles are equal. We call it very distinguished if in addition two of the other parts are equal. I.e. 5 sides and 2 angles or 2 sides and 5 angles.

   Show that every very distinguished pentagon has an axis of symmetry.