45. Österreichische Mathematik Olympiade

Gebietswettbewerb für Fortgeschrittene

1. April 2014

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1. Prove that there exist no positive real numbers x, y, z such that

$$\big(12x^2 + yz\big) \cdot \big(12y^2 + xz\big) \cdot \big(12z^2 + xy\big) = 2014x^2y^2z^2 \ .$$

2. Determine all quadruples (a, b, c, d) of real numbers satisfying the following system of equations.

$$\begin{aligned} ab + ac &= 3b + 3c \\ bc + bd &= 5c + 5d \\ ac + cd &= 7a + 7d \\ ad + bd &= 9a + 9b \end{aligned}$$

3. The sequence $\langle a_n \rangle$ is defined by the recursion

$$a_{n+1} = 5a_n^6 + 3a_{n-1}^3 + a_{n-2}^2 \qquad \text{for } n \geq 2$$

and the set of starting values $\{a_0, a_1, a_2\} = \{2013, 2014, 2015\}$.
(I.e., the starting values are these three numbers in *arbitrary* order.)

Show that the sequence does not contain any sixth power of an integer.

4. For a point P in the interior of a triangle ABC let D be the intersection of AP with BC, let E be the intersection of BP with AC and let F be the intersection of CP with AB.

Furthermore let Q and R be the intersections of the parallel to AB through P with the sides AC and BC, respectively. Likewise, let S and T be the intersections of the parallel to BC through P with the sides AB and AC, respectively.

In a given triangle ABC, determine all points P for which the triangles PRD, PEQ and PTE have the same area.

Working time: 4 hours.
Each problem is worth 8 points.