38^{th} Austrian Mathematical Olympiad

Beginner's Competition

June 14^{th}, 2007

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1. Prove that the number $9^n + 8^n + 7^n + 6^n - 4^n - 3^n - 2^n - 1^n$ is divisible by 10 for all integers $n \geq 0$.

   W. Janous, Innsbruck

2. Find all real solutions to the equation

   $$\lfloor x \rfloor^2 + \lfloor x \rfloor = x^2 - \frac{1}{4}.$$

   Here, $\lfloor x \rfloor$ denotes the largest integers less or equal to x.

   St. Wagner, Stellenbosch, South Africa

3. For real numbers $x \geq 0$ and $y \geq 0$, write $A = \frac{x+y}{2}$ for the arithmetic mean and $G = \sqrt{xy}$ for the geometric mean of x and y. Furthermore, let $W = \frac{\sqrt{x}+\sqrt{y}}{2}$ be the arithmetic mean of $\sqrt{x}$ and $\sqrt{y}$.

   Prove that

   $$G \leq W^2 \leq A.$$

   Determine all x and y with $G = W^2 = A$.

   R. Henner, Vienna

4. Consider a parallelogram $ABCD$ such that the midpoint M of the side CD lies on the angle bisector of $\measuredangle BAD$.

   Show that $\measuredangle AMB$ is a right angle.

   St. Wagner, Stellenbosch, South Africa