46^{th} Austrian Mathematical Olympiad

Beginners' Competition – Solutions

June 9, 2015

Problem 1. *Let a, b, c be integers with $a^3+b^3+c^3$ divisible by 18. Prove that abc is divisible by 6.*

(Karl Czakler)

Solution. We need to prove that abc is divisible by 2 and by 3. We will give proofs by contradiction.

Suppose that abc is odd. This implies that a, b and c are odd. Therefore, $a^3+b^3+c^3$ is odd and certainly not divisible by 18. This contradiction shows that abc is even.

Suppose that abc is not divisible by 3. Then neither a, b nor c is divisible by 3, i. e. they are in (possibly distinct) congruence classes among the following congruence classes mod 9.

x	1	2	4	-4	-2	-1
x^3	1	-1	1	-1	1	-1

We conclude that $a^3+b^3+c^3$ is equal to -3, -1, 1 or 3 mod 9. Therefore, $a^3+b^3+c^3$ is not divisible 9 and consequently not by 18. This contradiction shows that abc is divisible by 3.

(Gerhard Kirchner) □

Problem 2. *Let x, y be positive real numbers with $xy=4$.*

Prove that

$$\frac{1}{x+3}+\frac{1}{y+3}\le\frac{2}{5}.$$

For which x and y does equality hold?

(Walther Janous)

Solution. Clearing denominators, we obtain the equivalent inequality

$$5x+5y+30\le 2xy+6x+6y+18,$$

which simplifies to $x+y\ge 12-2xy=4$. This inequality is a direct consequence of the AM–GM inequality

$$\frac{x+y}{2}\ge\sqrt{xy}=2.$$

Equality holds exactly for $x=y=2$.

(Walther Janous) □

Problem 3. *Anton chooses as starting number an integer $n\ge 0$ which is not a square. Berta adds to this number its successor $n+1$. If this sum is a perfect square, she has won. Otherwise, Anton adds to this sum, the subsequent number $n+2$. If this sum is a perfect square, he has won. Otherwise, it is again Berta's turn and she adds the subsequent number $n+3$, and so on.*

Prove that there are infinitely many starting numbers, leading to Anton's win.

(Richard Henner)

Solution. We will prove that Anton wins for the infinity of starting numbers $3x^2-1$ with $x\ge 1$.

Since $3x^2-1\equiv 2 \bmod 3$, it cannot be a perfect square. After Berta adds the subsequent integer $3x^2$, the sum $6x^2-1$ is also $\equiv 2 \bmod 3$ and consequently not a perfect square. Now Anton adds the subsequent number $3x^2+1$ and obtains the perfect square $9x^2$. Therefore, Anton has won and we have found an infinity of possible starting numbers.

(Richard Henner) □

Problem 4. *Let k_1 and k_2 be internally tangent circles with common point X. Let P be a point lying neither on one of the two circles nor on the line through the two centers. Let N_1 be the point on k_1 closest to P and F_1 the point on k_1 that is farthest from P. Analogously, let N_2 be the point on k_2 closest to P and F_2 the point on k_2 that is farthest from P.*

Prove that $\angle N_1XN_2 = \angle F_1XF_2$.

(Robert Geretschläger)

Solution. The line segment N_1F_1 is a diameter of k_1 passing through P. Similarly, N_2F_2 is a diameter of k_2 passing through P.

Due to Thales's theorem, we have $\angle N_1XF_1 = 90\,°$ and $\angle N_2XF_2 = 90\,°$.

Letting $\angle N_2XF_1 = \alpha$, we obtain

$$\angle N_1XN_2 = 90\,° - \alpha \qquad \text{and} \qquad \angle F_1XF_2 = 90\,° - \alpha,$$

which proves the equality of the angles.

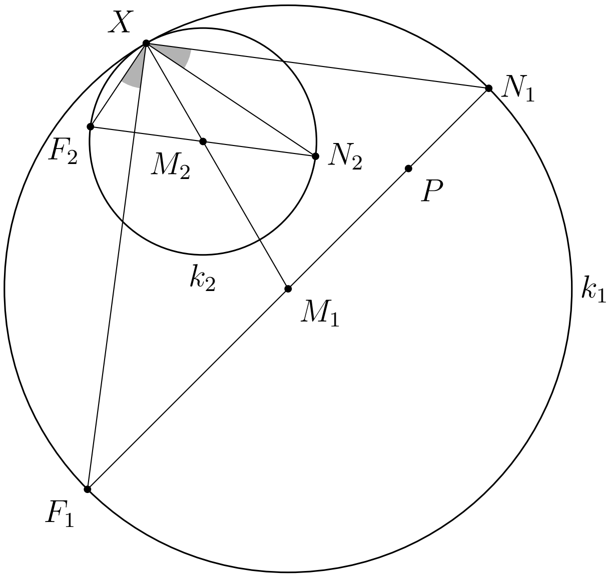

Figure 1: Problem 4

(Karl Czakler) □