46^{th} Austrian Mathematical Olympiad

Regional Competition (Qualifying Round)

March 26, 2015

---

1. Determine all triples (a, b, c) of positive integers satisfying the conditions

$$\gcd(a, 20) = b, \qquad \gcd(b, 15) = c \qquad \text{and} \qquad \gcd(a, c) = 5.$$

(Richard Henner)

2. Let x, y and z be positive real numbers with $x + y + z = 3$.
Prove that at least one of the three numbers

$$x(x + y - z), \qquad y(y + z - x) \qquad \text{or} \qquad z(z + x - y)$$

is less or equal 1.

(Karl Czakler)

3. Let $n \geq 3$ be a fixed integer. The numbers $1, 2, 3, \ldots, n$ are written on a board. In every move one chooses two numbers and replaces them by their arithmetic mean. This is done until only a single number remains on the board.

   Determine the least integer that can be reached at the end by an appropriate sequence of moves.

(Theresia Eisenkölbl)

4. Let ABC be an isosceles triangle with $AC = BC$ and $\angle ACB < 60^\circ$. We denote the incenter and circumcenter by I and O, respectively. The circumcircle of triangle BIO intersects the leg BC also at point $D \neq B$.

   (a) Prove that the lines AC and DI are parallel.

   (b) Prove that the lines OD and IB are mutually perpendicular.

(Walther Janous)

Working time: 4 hours.
Each problem is worth 8 points.