55^{th} Austrian Mathematical Olympiad

Regional Competition

21st March 2024

1. Let a, b and c be real numbers larger than 1. Prove the inequality

$$\frac{ab}{c-1} + \frac{bc}{a-1} + \frac{ca}{b-1} \geq 12.$$

When does equality hold?

(Karl Czakler)

2. Let ABC be an acute triangle with orthocenter H. The circumcircle of the triangle BHC intersects AC a second time in point P and AB a second time in point Q.

Prove that H is the circumcenter of the triangle APQ.

(Karl Czakler)

3. On a table, we have ten thousand matches, two of which are inside a bowl.

Anna and Bernd play the following game: They alternate taking turns and Anna begins. A turn consists of counting the matches in the bowl, choosing a proper divisor d of this number and adding d matches to the bowl. The game ends when more than 2024 matches are in the bowl. The person who played the last turn wins.

Prove that Anna can win independently of how Bernd plays.

(Richard Henner)

4. Let n be a positive integer.

Prove that $a(n) = n^5 + 5^n$ is divisible by 11 if and only if $b(n) = n^5 \cdot 5^n + 1$ is divisible by 11.

(Walther Janous)

Working time: 4 hours.
Each problem is worth 8 points.