55^{th} Austrian Mathematical Olympiad

Regional Competition—Solutions

21st March 2024

Problem 1. *Let a, b and c be real numbers larger than 1. Prove the inequality*

$$\frac{ab}{c-1}+\frac{bc}{a-1}+\frac{ca}{b-1}\geq 12.$$

When does equality hold?

(Karl Czakler)

Solution. By the AM-GM inequality, we know that

$$\sqrt{(c-1)\cdot 1}\leq \frac{c-1+1}{2},$$

therefore

$$c-1\leq \frac{c^2}{4}$$

with equality for $c=2$. With the two analogous inequalities for a and b we obtain

$$\frac{ab}{c-1}+\frac{bc}{a-1}+\frac{ca}{b-1}\geq \frac{4ab}{c^2}+\frac{4bc}{a^2}+\frac{4ca}{b^2}\geq 12\sqrt[3]{\frac{ab}{c^2}\cdot\frac{bc}{a^2}\cdot\frac{4ca}{b^2}}=12$$

where the last inequality is the AM-GM inequality again. Therefore, equality holds for $a=b=c=2$.

(Karl Czakler) □

Problem 2. *Let ABC be an acute triangle with orthocenter H. The circumcircle of the triangle BHC intersects AC a second time in point P and AB a second time in point Q.*

Prove that H is the circumcenter of the triangle APQ.

(Karl Czakler)

Solution. see Figure 1

Let H_a be the foot of the altitude on BC. With the angle sum in triangle AH_aC, we get

$$\angle HAC = 90^\circ - \angle BCA.$$

Let H_b be the foot of the altitude on AC. With the angle sum in triangle CH_bB, we get

$$\angle CBH = 90^\circ - \angle BCA.$$

The inscribed angle theorem gives us

$$\angle CPH = \angle CBH,$$

therefore

$$\angle CPH = \angle HAC.$$

We conclude that the triangle AHP is isosceles and we have $AH = PH$. Analogously, we can prove that $AH = QH$. Therefore, H is the circumcenter of the triangle APQ.

(Karl Czakler) □

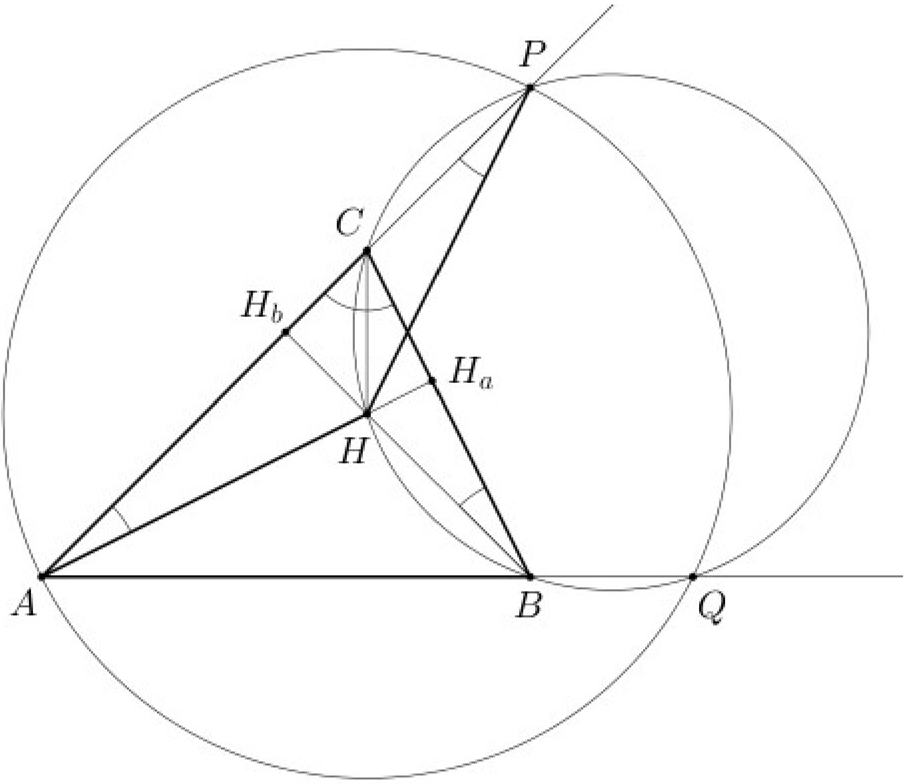

Figure 1: Problem 2

Problem 3. *On a table, we have ten thousand matches, two of which are inside a bowl.*

Anna and Bernd play the following game: They alternate taking turns and Anna begins. A turn consists of counting the matches in the bowl, choosing a proper divisor d of this number and adding d matches to the bowl. The game ends when more than 2024 matches are in the bowl. The person who played the last turn wins.

Prove that Anna can win independently of how Bernd plays.

(Richard Henner)

Solution. Anna's strategy consists of always adding a single match to the bowl while there are less than 1350 matches inside.

With this strategy, she will always change an even number of matches to an odd number of matches, so that Bernd is forced to choose an odd divisor and give her an even number of matches again.

Bernd will have at most 1350 matches and can add at most a third, since there is no larger odd proper divisor. Since $1350 + \frac{1}{3} \cdot 1350 = 1350 + 450 = 1800 < 2024$, he cannot reach more than 2024 matches in this phase of the game.

Therefore, there will come a turn where Anna starts with an even number of at least 1350 matches. She can add half of them and obtains at least $1350 + \frac{1}{2} \cdot 1350 = 1350 + 675 = 2025$ matches and has won.

(Richard Henner) □

Problem 4. *Let n be a positive integer.*

Prove that $a(n) = n^5 + 5^n$ is divisible by 11 if and only if $b(n) = n^5 \cdot 5^n + 1$ is divisible by 11.

(Walther Janous)

Solution. If n is a multiple of 11, both sides of the equivalence are wrong, so the equivalence is true.

If n is not a multiple of 11, Fermat's little theorem implies that $n^{10} - 1$ is a multiple of 11. The equivalence now follows from

$$n^5 a(n) = n^{10} + n^5 \cdot 5^n \equiv b(n) \pmod{11}.$$

(Gerhard Kirchner) □