50^{th} Austrian Mathematical Olympiad

National Competition—Final Round—Solutions

29th/30th May 2019

Problem 1. *Determine all functions* $f\colon \mathbb{R} \to \mathbb{R}$ *such that*

$$f(2x + f(y)) = x + y + f(x)$$

for all x, $y \in \mathbb{R}$.

(Gerhard Kirchner)

Answer. The only solution is $f(x) = x$ for all $x \in \mathbb{R}$.

Solution. We choose x so that the arguments on both sides become equal, i.e. the equation $2x + f(y) = x$ is satisfied. For this value $x = -f(y)$, we get $f(-f(y)) = -f(y) + y + f(-f(y))$ and therefore $f(y) = y$ for all $y \in \mathbb{R}$. But this is clearly a solution, therefore, it is the only solution.

(Gerhard Kirchner) □

Problem 2. *A (convex) trapezoid $ABCD$ shall be called* good *if it is inscribed, has parallel sides AB and CD, and CD is shorter than AB. For a good trapezoid, we fix the following notations.*

- *The line parallel to AD through B intersects the line CD in S.*
- *The tangents through S to the circumcircle of the trapezoid meet the circumcircle in E and F, respectively, where E is on the same side of the line CD as A.*

Characterize good trapezoids $ABCD$ (in terms of the side lengths and/or angles of the trapezoid) for which the angles $\angle BSE$ and $\angle FSC$ are equal. The characterization should be as simple as possible.

(Walther Janous)

Answer. The angles $\angle BSE$ and $\angle FSC$ are equal if and only if $\angle BAD = 60°$ or $AB = AD$.

Solution. We denote the circumcircle of the trapezoid by u, the second intersection point of the line SB with u by T and the centre of u by M, see Figure 1. As the trapezoid is inscribed, it is isosceles. As $ABSD$ is a parallelogram by construction, we have $BS = AD = BC$ and $DS = AB$.

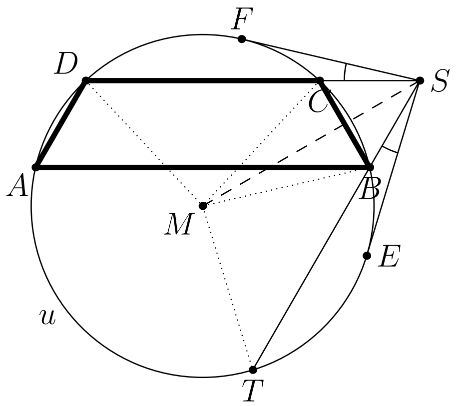

Figure 1: Problem 2, Case 1: B between S and T

Consider the reflection across the line MS. It clearly maps E and F to each other and maps u to itself. We say that the trapezoid meets the *angle condition* if $\angle BSE = \angle FSC$.

The trapezoid meets the angle condition if and only if the reflection maps the rays SB and SC to each other. Equivalently, the intersection points of these rays with u are mapped to each other corresponding to the order of the points on the rays.

We first consider the case that B is between S and T, see Figure 1. Then the trapezoid meets the angle condition if and only if the reflection maps B and C to each other. Equivalently, the triangle BSC is isosceles with axis of symmetry SM. As M lies on the perpendicular bisector of BC in any case, this is equivalent to $CS = BS$. As $BS = BC$, this is in turn equivalent to the triangle BSC being equilateral. Again by $BS = BC$, this is equivalent to $\angle CSB = 60°$. As $ABSD$ is a parallelogram, the trapezoid meets the angle condition in this case if and only if $\angle BAD = 60°$.

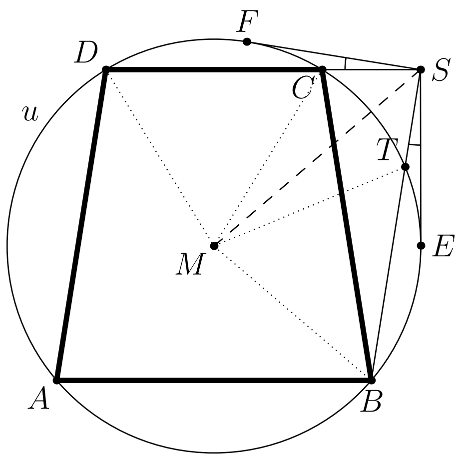

Figure 2: Problem 2, Case 2: T between S and B

We now consider the case that T lies between S and B, see Figure 2. Then the above considerations show that the trapezoid meets the angle condition if and only if the reflection maps B and D to each other. Equivalently, the triangle BSD is isosceles with axis of symmetry MS. By the same argument as in the first case, this is equivalent to $SB = SD$. This is equivalent to $AB = AD$.

(Clemens Heuberger) □

Problem 3. *In the country of Oddland, there are stamps with values 1 cent, 3 cent, 5 cent, etc., one type for each odd number. The rules of Oddland Postal Services stipulate the following: for any two distinct values, the number of stamps of the higher value on an envelope must never exceed the number of stamps of the lower value.*

In the country of Squareland, on the other hand, there are stamps with values 1 cent, 4 cent, 9 cent, etc., one type for each square number. Stamps can be combined in all possible ways in Squareland without additional rules.

Prove for every positive integer n: In Oddland and Squareland there are equally many ways to correctly place stamps of a total value of n cent on an envelope. Rearranging the stamps on an envelope makes no difference.

(Stephan Wagner)

Solution. We construct a bijection between possible combinations in Oddland and possible combinations in Squareland. Suppose we have a combination of Squareland stamps that sum to n cent, consisting of a_1 stamps of value 1 cent, a_2 stamps of value 4 cent, ..., a_M stamps of value M^2 cent, so that

$$n = \sum_{k=1}^{M} k^2 a_k.$$

Now we express k^2 as $\sum_{j=1}^{k}(2j-1)$ and interchange the order of summation, which yields

$$n = \sum_{k=1}^{M}\sum_{j=1}^{k}(2j-1)a_k = \sum_{j=1}^{M}(2j-1)\sum_{k=j}^{M}a_k.$$

This gives us a possible combination of Oddland stamps: By setting $b_j = \sum_{k=j}^{M} a_k$, we have

$$n = \sum_{j=1}^{M}(2j-1)b_j.$$

This can be interpreted as a collection of b_1 stamps of value 1 cent, b_2 stamps of value 3 cent, $\ldots$, b_M stamps of value $(2M-1)$ cent. We have $b_1 \geq b_2 \geq \cdots \geq b_M$ by definition, so this is a legal combination in Oddland.

Conversely, if a combination in Oddland is given by the values $b_1, b_2, \ldots, b_M$, we can use the identities $a_1 = b_1 - b_2$, $a_2 = b_2 - b_3$, $\ldots$, $a_{M-1} = b_{M-1} - b_M$, $a_M = b_M$ to recover the corresponding combination in Squareland. (Note that these values are nonnegative whenever $b_1 \geq b_2 \geq \cdots \geq b_M$.)

Since these two operations obviously are inverse to one another, we have found a bijection, which proves the statement.

(Stephan Wagner) □

Problem 4. *Let a, b and c be positive real numbers satisfying $a + b + c + 2 = abc$.*

Prove

$$(a+1)(b+1)(c+1) \geq 27.$$

When does equality occur?

(Karl Czakler)

Answer. Equality occurs if and only if $a = b = c = 2$.

Solution. We set $x = a + 1$, $y = b + 1$ and $z = c + 1$. Thus we have to show

$$xyz \geq 27$$

subject to

$$xyz = xy + yz + zx.$$

From the constraint we get

$$xyz = xy + yz + zx \geq 3\sqrt[3]{x^2y^2z^2}$$

by using the inequality between the arithmetic and the geometric means of xy, yz and zx. This is clearly equivalent to $xyz \geq 27$.

Equality occurs if and only if $xy = yz = zx$, or, equivalently, $x = y = z$. By the constraint, this is equivalent to $x = y = z = 3$ and finally $a = b = c = 2$.

(Clemens Heuberger) □

Problem 5. *We are given an arbitrary acute-angled triangle ABC and its altitudes AD and BE where D and E denote their feet on sides BC and AC, respectively. Let furthermore F and G be two points on segments AD and BE, respectively, such that*

$$\frac{AF}{FD} = \frac{BG}{GE}.$$

The line through C and F intersects BE in point H and the line through C and G intersects AD in point I. Prove that the four points F, G, H and I are concyclic.

(Walther Janous)

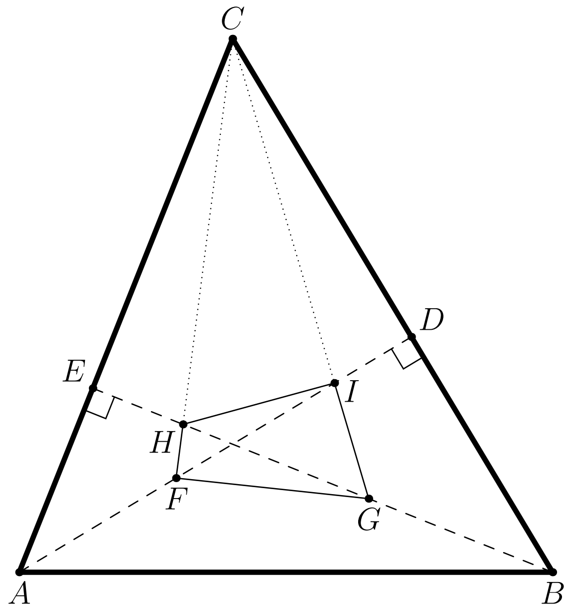

Figure 3: Problem 5

Solution. The two right-angled triangles ADC and BEC are inversely similar to each other, see Figure 3. Here, the sides AD and BE correspond to each other.

But the condition

$$\frac{AF}{FD} = \frac{BG}{GE}$$

means: The two points F and G divide the two sides AD and BE, respectively, in equal ratios. Thus, the two oriented angles $\angle DFC$ and $\angle CGE$ are equal, which implies that the oriented angles $\angle IFH$ and $\angle IGH$ are equal modulo 180°. Thus the inscribed angle theorem implies that the four points F, G, H and I are concyclic.

(Walther Janous) □

Remark. The solution only uses that $ABDE$ is inscribable.

Problem 6. *Determine the smallest possible positive integer n with the following property: For all positive integers x, y and z with $x \mid y^3$ and $y \mid z^3$ and $z \mid x^3$ we also have $xyz \mid (x+y+z)^n$.*

(Gerhard J. Woeginger)

Answer. The smallest possible integer with that property is $n = 13$.

Solution. We note that we have $xyz \mid (x+y+z)^n$ if and only if for each prime p the inequality $v_p(xyz) \le v_p((x+y+z)^n)$ holds, where as usual $v_p(m)$ denotes the exponent of p in the prime factorization of m.

Let x, y and z be positive integers with $x \mid y^3$, $y \mid z^3$ and $z \mid x^3$. Let p be an arbitrary prime, and w.l.o.g. let the multiplicity of p be lowest in z, that is, $v_p(z) = \min\{v_p(x), v_p(y), v_p(z)\}$.

Then we have $v_p(x+y+z) \ge v_p(z)$, and from the divisibility constraints we get $v_p(x) \le 3v_p(y) \le 9v_p(z)$. It follows that

$$\begin{aligned} v_p(xyz) &= v_p(x) + v_p(y) + v_p(z) \\ &\le 9v_p(z) + 3v_p(z) + v_p(z) = 13v_p(z) \\ &\le 13v_p(x+y+z) = v_p((x+y+z)^{13}), \end{aligned} \tag{1}$$

which proves that for $n = 13$ the desired property is satisfied.

It remains to show that this is indeed the smallest possible integer with this property. For doing so, let n now be a number that has the desired property. By setting $(x, y, z) = (p^9, p^3, p^1)$ with an arbitrary prime p (in order to achieve that both inequalities in (1) become equalities), we get

$$\begin{aligned}13 &= v_p(p^{13}) = v_p(p^9 \cdot p^3 \cdot p^1) = v_p(xyz)\\ &\leq v_p((x + y + z)^n) = v_p((p^9 + p^3 + p^1)^n) = n \cdot v_p(p(p^8 + p^2 + 1)) = n,\end{aligned}$$

which yields $n \geq 13$.

(Birgit Vera Schmidt, Gerhard J. Woeginger) □