51^{st} Austrian Mathematical Olympiad

Regional Competition—Solutions

2nd April 2020

Problem 1. *Determine all positive integers a for which the equation*

$$\left(1+\frac{1}{x}\right)\cdot\left(1+\frac{1}{x+1}\right)\cdots\left(1+\frac{1}{x+a}\right)=a-x$$

has at least one integer solution x.

For each such integer a, determine the corresponding solutions.

(Richard Henner)

Answer. Only $a=7$ yields at least one integer solution. In this case, the solutions are $x=2$ and $x=4$.

Solution. The left-hand side of the equation is

$$\left(1+\frac{1}{x}\right)\cdot\left(1+\frac{1}{x+1}\right)\cdots\left(1+\frac{1}{x+a}\right)=\frac{x+1}{x}\cdot\frac{x+2}{x+1}\cdots\frac{x+a+1}{x+a}=\frac{x+a+1}{x}.$$

Hence for $x\notin\{0,-1,\ldots,-a\}$ the equation is equivalent to $x^2+(1-a)x+a+1=0$. The roots of this quadratic equation are $x=\frac{a-1\pm\sqrt{a^2-6a-3}}{2}$.

For $0<a\le 6$, we have $a^2-6a=a\cdot(a-6)\le 0$ and therefore, $a^2-6a-3<0$. Hence the equation has no real roots.

For $a>9$, we have $(a-4)^2<a^2-6a-3<(a-3)^2$ and therefore, the roots cannot be integers.

Thus we only have to consider the cases $a=7$, $a=8$ and $a=9$. There are no integer roots for the cases $a=8$ and $a=9$. For $a=7$ we get $x=2$ and $x=4$.

(Richard Henner) □

Problem 2. *The set M consists of all 7-digit positive integers which contain each of the digits 1, 3, 4, 6, 7, 8 and 9 (in base 10) exactly once.*

a) Determine the smallest positive difference d between any two numbers in M.

b) How many pairs (x,y) with x and y in M exist for which $x-y=d$ holds?

(Gerhard Kirchner)

Answer. a) The smallest difference is 9. b) There exist 480 pairs.

Solution. a) For all numbers in the set M, the sum of digits is $1+3+4+6+7+8+9=38$. Hence they all lie in the same residue class modulo 9. The difference d is therefore a multiple of 9 and as a consequence $d\ge 9$. As an example, the two numbers $x=1346798$ and $y=1346789$ fulfil the equation $x-y=9$, and hence $d=9$ is the smallest possible difference.

b) The equation $x=y+9$ yields the following possibilities for the units digits of the two numbers:

units digit of y	1	3	4	6	7	8	9
units digit of x	–	–	3	–	6	7	8

When adding 9 to the units digit 4, 7, 8 or 9 of y, we get a carry of 1, which is added to the tens digit. We know that there are no zeroes among the digits of x, so the tens digit of y cannot be 9. Therefore there is no further carry and the rest of the digits of the two numbers have to coincide. Thus y has to end with the two digits 34, 67, 78 or 89 and x with the two digits 43, 76, 87 or 98, respectively.

Thus, there are 4 possibilities for the last two digits of y and $5! = 120$ possibilities for the remaining digits. Each of these numbers y has exactly one corresponding number x, and hence there are 480 such pairs.

(Gerhard Kirchner) □

Problem 3. *Let ABC be a triangle with $AB < AC$ and incenter I. The perpendicular bisector of the side BC intersects the angle bisector of $\angle BAC$ at the point S, and the angle bisector of $\angle CBA$ at the point T, respectively.*

Show that the points C, I, S and T lie on a common circle.

(Karl Czakler)

Solution. The problem is illustrated in Figure 1.[1] Let α and β be the angles of the triangle in A and B, respectively.

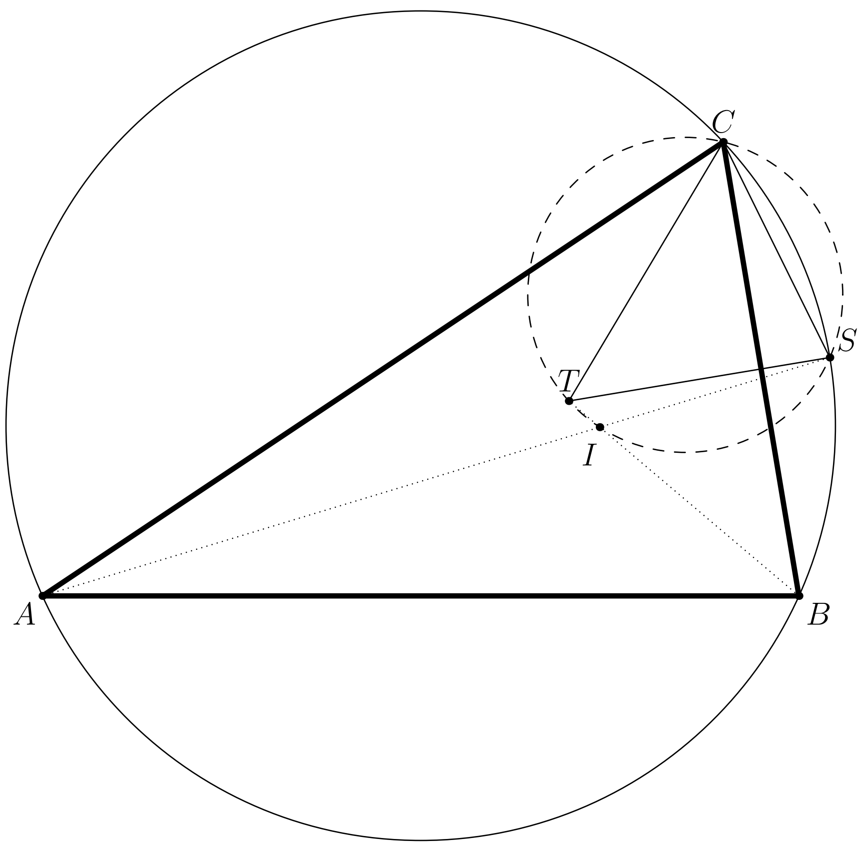

Figure 1: Problem 3

We have

$$\angle SIT = \angle AIB = 180^\circ - \frac{\alpha + \beta}{2}.$$

[1]Note that this is the only possible configuration: As $AC > AB$, the points A and C lie in different half planes with respect to the perpendicular bisector of BC. Thus the segment AS and C do not lie in the same half plane with respect to this bisector. As S lies on the circumcircle of the triangle ABC and I lies in the interior of the triangle, we conclude that I and C lie in different half planes with respect to the bisector of BC.

It is well known that the common point S of the angle bisector in A and the bisector of the side BC lies on the circumcircle of $\triangle ABC$. By the inscribed angle theorem,

$$\angle BCS = \angle BAS = \frac{\alpha}{2}.$$

As T lies on the bisector of the side BC, it follows that

$$\angle TCB = \angle CBT = \frac{\beta}{2},$$

and thus

$$\angle TCS = \frac{\alpha + \beta}{2}.$$

Opposite angles of the quadrilateral $TISC$ sum to 180° and hence $TISC$ is a cyclic quadrilateral.

(Karl Czakler) □

Problem 4. *Determine all quadruples* (p, q, r, n) *which satisfy the equation*

$$p^2 = q^2 + r^n$$

where p, q, r *are prime numbers and* n *is a positive integer.*

(Walther Janous)

Answer. Exactly the two quadruples $(3, 2, 5, 1)$ and $(5, 3, 2, 4)$ fulfill the requirement.

Solution. The equation is equivalent to

$$(p - q)(p + q) = r^n.$$

We consider two cases:

(a) $p - q = 1$ and hence $p = 3$, $q = 2$ and $r^n = 5$. It follows immediately that $r = 5$ and $n = 1$, and we obtain the quadruple $(3, 2, 5, 1)$ as the first solution.

(b) $p - q > 1$. The inequality $p + q > p - q$ requires $n \geq 2$ and furthermore $r \mid p - q$ and $r \mid p + q$. This yields $r \mid 2p$ and $r \mid 2q$ as r is a divisor of the sum and the difference of $(p + q)$ and $(p - q)$, respectively.

Assume that $r \neq 2$. Then $r = p$ and $r = q$ and hence $r^n = 0$, which is a contradiction to r being a prime number.

Therefore $r = 2$ and there exist $1 \leq a < b$ with $a + b = n$ and

$$p - q = 2^a \quad \text{and} \quad p + q = 2^b.$$

Then $p - q + p + q = 2p = 2^a + 2^b$ and equivalently $p = 2^{a-1}(2^{b-a} + 1)$.

The inequality $2^{b-a} + 1 \geq 2^1 + 1 = 3$ yields $a = 1$ and thus $p = 2^{b-1} + 1$ and hence $p - q = 2$ and $q = 2^{b-1} - 1$. Thus $q = 2^{b-1} - 1$, 2^{b-1} and $p = 2^{b-1} + 1$ are three consecutive integers, one of which is certainly divisible by 3. As p and q are primes and the middle number is a power of 2, we have $q = 3$ and $p = 5$ (the alternative being $q = 1$ and $p = 3$, which is impossible). It follows immediately that $n = 4$ and we get the quadruple $(5, 3, 2, 4)$ as the second solution.

(Walther Janous) □