51^{st} Austrian Mathematical Olympiad

National Competition—Final Round—Solutions

20th & 27th June 2020

Problem 1. *Let $ABCD$ be a cyclic quadrilateral and let S be the intersection point of its diagonals. Furthermore let P be the circumcenter of the triangle ABS and Q the circumcenter of the triangle BCS. The parallel to AD through P and the parallel to CD through Q intersect in point R.*

Prove that R is on BD.

(Karl Czakler)

Solution. We use directed angles modulo 180°. Let E be the foot of S on CD, see Figure 1.

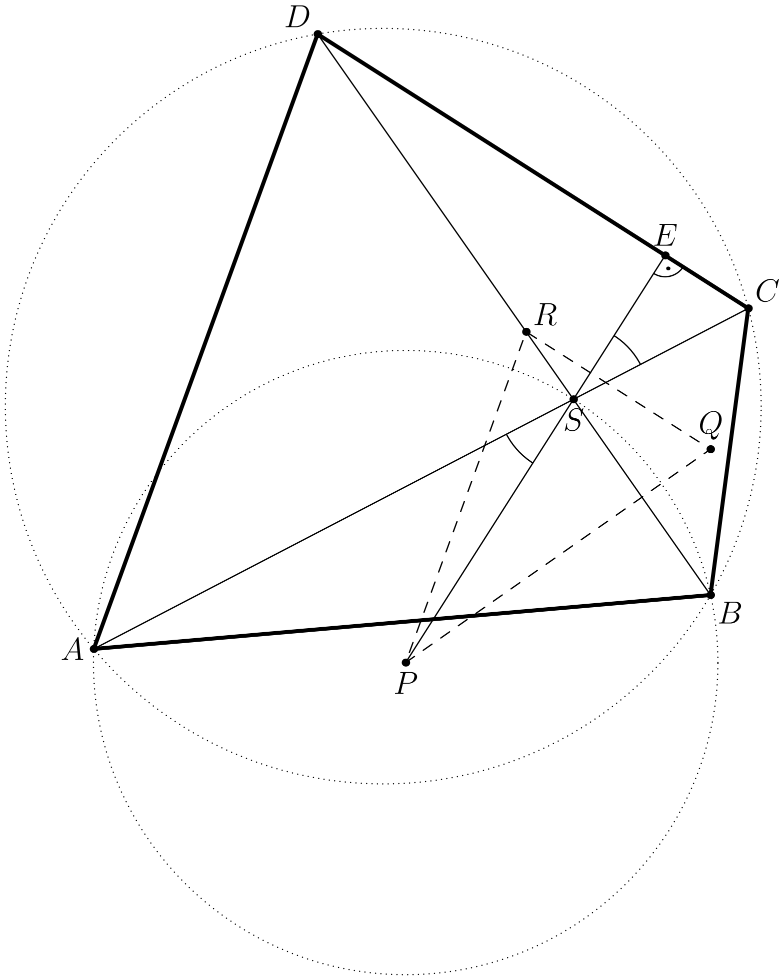

Figure 1: Problem 1

Due to the sum of angles in the isosceles triangle APS and the central angle theorem in the circumcircle of the triangle ABS, we get

$$\angle ASP = 90^\circ - \frac{1}{2}\angle SPA = 90^\circ - \angle SBA.$$

The inscribed angle theorem in the cyclic quadrilateral $ABCD$ yields

$$90^\circ - \angle SBA = 90^\circ - \angle DBA = 90^\circ - \angle DCA.$$

Using the sum of angles in the right triangle CES, we finally get

$$90^\circ - \angle DCA = \angle CSE.$$

Combining this, we get $\angle ASP = \angle CSE$.

Therefore, the points P, S and E are collinear. The line through them is perpendicular to DC and therefore also to its parallel QR. The altitude of the triangle PQR through the vertex P therefore is part of the line PS. Analogously, we show that the altitude of PQR through Q is part of the line QS.

Therefore, S is the orthocenter of the triangle PQR. Since the line BD is perpendicular to its bisector PQ and passes through S, the point R must be on that line.

(Karl Czakler) □

Problem 2. *There are* 2020 *points in the plane, some of which are black and the others are green.*

For each black point the following holds: There are exactly two green points that have a distance of 2020 *to this black point.*

Determine the smallest possible number of green points.

(Walther Janous)

Answer. The smallest possible number is 45 green points.

Solution. We define *green circles* as circles with a green center and with a radius of 2020. According to the problem statement, black points can only be placed in intersection points of exactly two green circles.

If we have g green circles, each can intersect with each other at most twice, so there can be at most $\binom{g}{2} \cdot 2 = g^2 - g$ black points. Together with the g green points, we have a maximum of $g^2 - g + g = g^2$ points.

Due to $44^2 < 2020 (< 45^2)$, we therefore need at least 45 green points.

There are numerous ways to place 45 green circles in such a way that each intersects with each other twice and there is no point where more than two of them intersect.

For example, we can choose an arbitrary line segment AB of length less than 4040 anywhere in the plane and place the 45 green points on it in any way we wish.

It is clear that any two green circles intersect twice because the distance of their centers is less than twice their radius.

Furthermore, it is easy to check that no three green circles intersect in one point, because the further apart the centers of two intersecting circles, the smaller the orthogonal distance of their intersection point to AB. If there existed a point S where three circles intersect, then their centers would have to satisfy $M_1M_2 = M_2M_3 = M_1M_3$, which is not possible because they are all on the same line segment.

We therefore have $\frac{45 \cdot (45-1)}{2} \cdot 2 = 1980$ points in which two of the 45 circles intersect, and place black points in 1975 of them.

This shows that a configuration with 45 green points is indeed possible and concludes the proof.

(Walther Janous, Birgit Vera Schmidt) □

Problem 3. *Let a be a fixed positive integer and (e_n) the sequence defined by $e_0 = 1$ and*

$$e_n = a + \prod_{k=0}^{n-1} e_k$$

for $n \geq 1$.

(a) *Prove that there are infinitely many primes that divide an element of the sequence.*

(b) *Prove that there exists a prime that divides no element of the sequence.*

(Theresia Eisenkölbl)

Solution.

Lemma. The equation $\gcd(a, e_n) = 1$ holds for all $n \geq 0$.

Proof. We show by induction on n that $\gcd(a, e_k) = 1$ for all $k \leq n$. This is obvious for $n = 0$. For the induction step, we note that $\gcd(a, e_n) = \gcd(a, a + \prod_{k=0}^{n-1} e_k) = \gcd(a, \prod_{k=0}^{n-1} e_k) = 1$, which proves the lemma. ∎

For the first part, we now follow the strategy of Euclid's proof that there are infinitely many primes. Let $p_1, \ldots, p_N$ be distinct primes which each divide at least one element of the sequence. We can choose M so that each of these primes divides an e_n with $n < M$, which implies that all of them divide $\prod_{k=0}^{M-1} e_k$. By the lemma, they cannot divide a, therefore, none of these primes can divide e_M. Since $e_M > 1$, it must have a prime divisor that is not yet on our list. Therefore, we can always add new prime divisors to our list and find an infinite number of primes which divide an element of the given sequence.

For the second part, we start with the case $a \neq 1$. We choose one of the primes that divide a. By the lemma, this prime divides no element of the sequence as desired.

From now on, we assume $a = 1$ and we define $m_n := \prod_{k=0}^{n} e_k$ for $n \geq 0$. From the given recurrence relation, we immediately get $m_{n+1} = m_n e_{n+1} = m_n(1 + m_n)$ and $m_0 = 1$. Modulo 5, we get $m_1 \equiv 2 \pmod 5$, $m_2 \equiv 1 \pmod 5$. Since m_{n+1} only depends on m_n, we see that m_n is always congruent to 1 or 2 modulo 5 and therefore never divisible by 5. Its divisor e_n is also never divisible by 5 as desired.

(Clemens Heuberger) □

Problem 4. *Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying*

$$f(xf(y) + 1) = y + f(f(x)f(y))$$

for all x, $y \in \mathbb{R}$.

(Theresia Eisenkölbl)

Answer. There is one such function, namely $f(x) = x - 1$, $x \in \mathbb{R}$.

Solution. We prove injectivity by setting both $y = a$ and $y = b$ with $f(a) = f(b)$ in the original equation and obtaining

$$a = f(xf(a) + 1) - f(f(x)f(a)) = f(xf(b) + 1) - f(f(x)f(b)) = b.$$

Now, we set $y = 0$. Using injectivity, we obtain

$$xf(0) + 1 = f(x)f(0).$$

For $f(0) = 0$, this becomes $1 = 0$, which is not possible. Therefore, we can divide by $f(0)$ and obtain $f(x) = x + c$ for some constant c.

Plugging this into the original equation immediately gives that the only solution is $c = -1$. Therefore, the only solution is $f(x) = x - 1$.

(Theresia Eisenkölbl) □

Problem 5. *Let h be a semicircle with diameter AB. An arbitrary point P on the line segment AB is chosen. The line through P that is perpendicular to AB intersects h at point C. The line segment PC divides the area of the semicircle into two parts. In each of them, a circle is inscribed that touches AB, PC and h. The point of tangency of AB and each circle is D and E, respectively, where D is between A and P.*

Prove that the size of the angle $\angle DCE$ does not depend on the choice of P.

(Walther Janous)

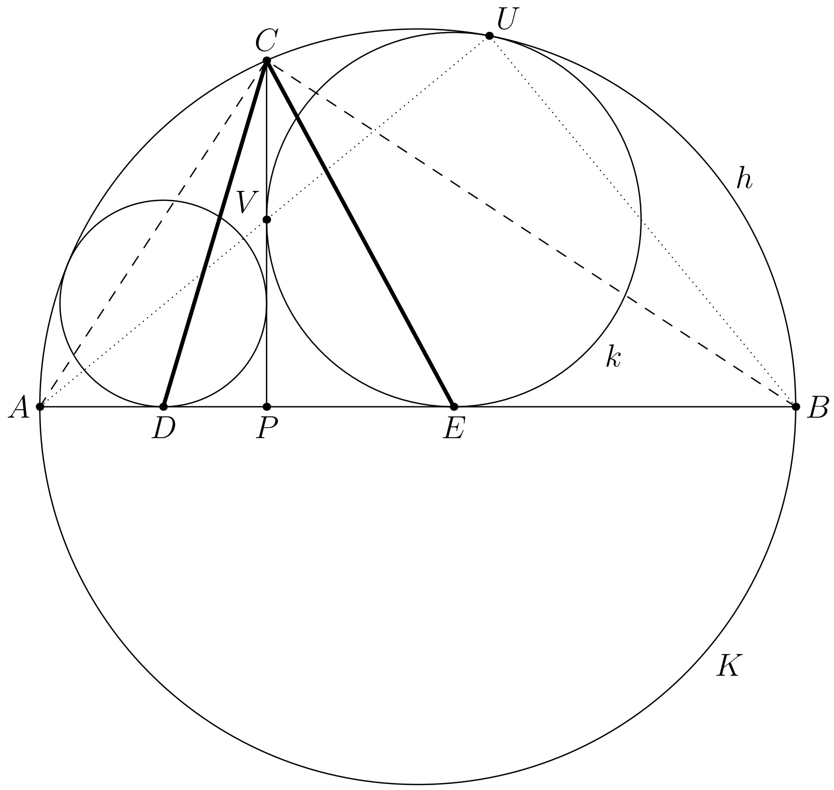

Figure 2: Problem 5

Solution. We denote the inscribed circle that touches AB in E by k. Furthermore, we denote the point of tangency of k and h by U and the point of tangency of k and PC by V, see Figure 2. Finally, let K be the circle of which h is one half.

Lemma 1. The points U, V and A are collinear.

Proof. The point of tangency U of k and K is the center of a homothety that maps k to K. Therefore, also the point of tangency of the tangent perpendicular to AB is mapped to the point of tangency with the tangent perpendicular to AB, that is, V is mapped to A. Hence U, V and A are collinear. ∎

Lemma 2. It holds that $AC = AE$.

Proof. From Lemma 1 we get that the triangles APV and AUB are similar, because they have one angle in common and both have one right angle. Therefore, we have $\frac{AV}{AP} = \frac{AB}{AU}$ and thus $AV \cdot AU = AP \cdot AB$.

Using the power of the point A with respect to the circle k, we get

$$AE^2 = AV \cdot AU = AP \cdot AB.$$

On the other hand, it is well known that in a right triangle, the length of a cathetus is the geometric mean of the length of the adjacent segment cut by the altitude to the hypotenuse and the length of the whole hypotenuse (Euclid's theorem); applying this in triangle ABC yields that $AC^2 = AP \cdot AB$. Therefore, we have $AC = AE$ as desired. ∎

In the isosceles triangle EAC it therefore follows that

$$\angle PCE = 90^\circ - \angle AEC = \angle CAB/2 = (90^\circ - \angle ABC)/2 = \angle PCB/2.$$

This means that the line CE is the angle bisector of $\angle PCB$. Analogously, CD is the angle bisector of $\angle ACP$.

This implies $\angle DCE = \angle ACB/2 = 45^\circ$ independent of the choice of P.

(Theresia Eisenkölbl) □

Problem 6. *The players Alfred and Bertrand together determine a polynomial $x^n + a_{n-1}x^{n-1} + \cdots + a_0$ of the given degree $n \geq 2$. To do so, in n moves they alternatingly choose the value of one coefficient, where all coefficients must be integers and $a_0 \neq 0$ must hold. Alfred makes the first move. Alfred wins if the final polynomial has an integer root.*

(a) For which n is Alfred able to force a victory if the coefficients a_j are chosen from right to left, that is, for $j = 0, 1, \ldots, n-1$?

(b) For which n is Alfred able to force a victory if the coefficients a_j are chosen from left to right, that is, for $j = n-1, n-2, \ldots, 0$?

(Theresia Eisenkölbl, Clemens Heuberger)

Answer. In both cases, Alfred can force a victory if and only if n is odd.

Solution. (a) If Alfred makes the last move, he sets $x = 1$ and gets a linear equation in a_{n-1} for making 1 a root. Since a_{n-1} has a coefficient of 1 in this linear equation, it has an integer solution.

If Bertrand makes the last move, it is known that only divisors of the absolute term are candidates for roots. He therefore gets a finite number of linear equations in a_{n-1} for one of these divisors being a root and chooses a_{n-1} in such a way that it does not satisfy any of them.

(b) If Alfred makes the last move, he writes $P(x) = xQ(x) + a_0$. He chooses an integer $y \neq 0$ with $Q(y) \neq 0$ (which is certainly possible because $Q(x)$ is not the zero polynomial) and chooses $a_0 = -yQ(y)$. This ensures that $a_0 \neq 0$ and $P(y) = 0$.

If Bertrand makes the last move, he also writes

$$P(x) = xQ(x) + a_0$$

and chooses a prime p that is neither one of the finitely many roots of $Q(x)+1 = 0$ or $Q(-x)-1 = 0$ nor is equal to $-Q(1)$ or $Q(-1)$. He then sets $a_0 = p$. The conditions ensure that $P(\pm p) \neq 0$ and $P(\pm 1) \neq 0$ hold and therefore P cannot have any integer roots.

This means that Alfred can force a victory if and only if he makes the last move, that is, if n is odd.

(Clemens Heuberger) □