CAPS Match 2024: Problems

ISTA, Austria
June 30 – July 3, 2024

Problem 1. Determine whether there exist 2024 distinct positive integers satisfying the following: If we consider every possible ratio between two distinct numbers (we include both a/b and b/a), we will obtain numbers with finite decimal expansions (after the decimal point) of mutually distinct non-zero lengths. (Patrik Bak, Slovakia)

Problem 2. For a positive integer n, an *n-configuration* is a family of sets $\langle A_{i,j}\rangle_{1\le i,j\le n}$. An n-configuration is called *sweet* if for every pair of indices (i, j) with $1 \le i \le n-1$ and $1 \le j \le n$ we have $A_{i,j} \subseteq A_{i+1,j}$ and $A_{j,i} \subseteq A_{j,i+1}$. Let $f(n, k)$ denote the number of sweet n-configurations such that $A_{n,n} \subseteq \{1, 2, \dots, k\}$. Determine which number is larger: $f(2024, 2024^2)$ or $f(2024^2, 2024)$. (Wojciech Nadara, Poland)

Problem 3. Let ABC be a triangle and D a point on its side BC. Points E, F lie on the lines AB, AC beyond vertices B, C, respectively, such that $BE = BD$ and $CF = CD$. Let P be a point such that D is the incenter of triangle PEF. Prove that P lies inside the circumcircle Ω of triangle ABC or on it. (Josef Tkadlec, Czech Republic)

Problem 4. Let $ABCD$ be a quadrilateral, such that $AB = BC = CD$. There are points X, Y on rays CA, BD, respectively, such that $BX = CY$. Let P, Q, R, S be the midpoints of segments BX, CY, XD, YA, respectively. Prove that points P, Q, R, S lie on a circle. (Michal Pecho, Slovakia)

Problem 5. Let $\alpha \neq 0$ be a real number. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 + y^2) = f(x - y)f(x + y) + \alpha y f(y)$$

holds for all x, $y \in \mathbb{R}$. (Walther Janous, Austria)

Problem 6. Determine whether there exist infinitely many triples (a, b, c) of positive integers such that p divides $\lfloor (a + b\sqrt{2024})^p \rfloor - c$ for every prime p.

Note: $\lfloor x \rfloor$ denotes the largest integer not larger than x. (Walther Janous, Austria)