52nd Austrian Mathematical Olympiad

Junior Regional Competition—Solutions

15th June 2021

Problem 1. *The pages of a notebook are numbered consecutively such that the first sheet contains the numbers* 1 *and* 2*, the second sheet contains the numbers* 3 *and* 4*, and so on. One sheet is torn out of the notebook. The page numbers on the remaining sheets are added. The resulting sum equals* 2021*.*

(a) How many pages can the notebook have had originally?

(b) Which page numbers could be found on the sheet that has been torn out?

(Walther Janous)

Answer. There is exactly one solution. The notebook had 64 pages and the sheet with the page numbers 29 and 30 is ripped out.

Solution. Let $b > 0$ be the number of sheets. The number of pages will be $2b$. We are looking for a number $2b$ such that

$$1 + 2 + \cdots + (2b - 1) + 2b = \frac{(2b) \cdot (2b + 1)}{2} > 2021.$$

Since $\frac{60^2}{2} = 1800$ has the right order of magnitude, we check the integers beginning with $b = 30$ and find:

$$\frac{62 \cdot 63}{2} = 1953 < 2021 < \frac{64 \cdot 65}{2} = 2080.$$

Therefore, the smallest possible number of pages is 64.

The torn out sheet h contains the page numbers $2h - 1$ and $2h$ (first odd, then even).

This gives the equation

$$2h - 1 + 2h = 2080 - 2021 = 59$$

which implies $h = 15$.

So, one solution is that the book originally had 32 sheets and the 15th sheet with page numbers 29 and 30 has been torn out.

It remains to explain why this is the only solution. If the book has 64 sheets, this is the only possibility because h could be computed uniquely. Now, assume that the number b of sheets is larger than 32 and the number of pages at least 66.

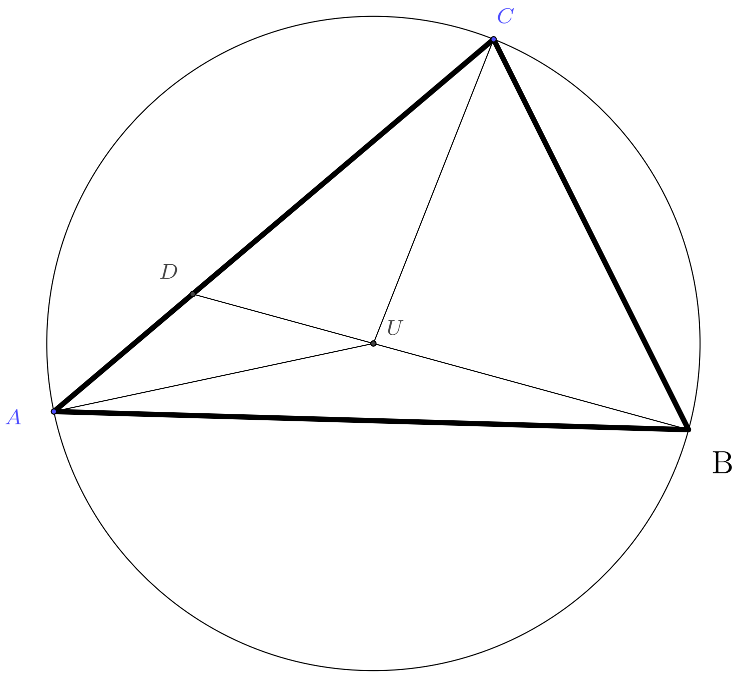
The sheet that has been torn out can have at most page numbers $2b - 1$ and $2b$, so the remaining sum is at least $1 + 2 + \cdots + 63 + 64 = 2080 > 2021$. So there cannot be a solution with more than 32 sheets.

(Lukas Donner) □

Problem 2. *Let ABC be a triangle with circumcenter U such that $\angle CBA = 60°$ and $\angle CBU = 45°$. Let D be the point of intersection of the lines BU and AC.*

Prove that $AD = DU$.

(Karl Czakler)



Solution.

In the isosceles triangle AUB, we have

$$\angle BAU = \angle UBA = 60^\circ - 45^\circ = 15^\circ,$$

and therefore

$$\angle AUB = 180^\circ - \angle BAU - \angle UBA = 150^\circ.$$

The inscribed angle theorem implies

$$\angle BCA = \frac{1}{2}\angle BUA = 75^\circ,$$

and therefore

$$\angle BAC = 180^\circ - 60^\circ - 75^\circ = 45^\circ.$$

We can finally compute the two angles of interest:

$$\angle UAD = \angle BAD - \angle BAU = \angle BAC - \angle BAU = 45^\circ - 15^\circ = 30^\circ$$
$$\angle DUA = 180^\circ - \angle AUB = 180^\circ - 150^\circ = 30^\circ$$

Therefore, the triangle AUD is isosceles with apex D and we have $AD = DU$ as desired.

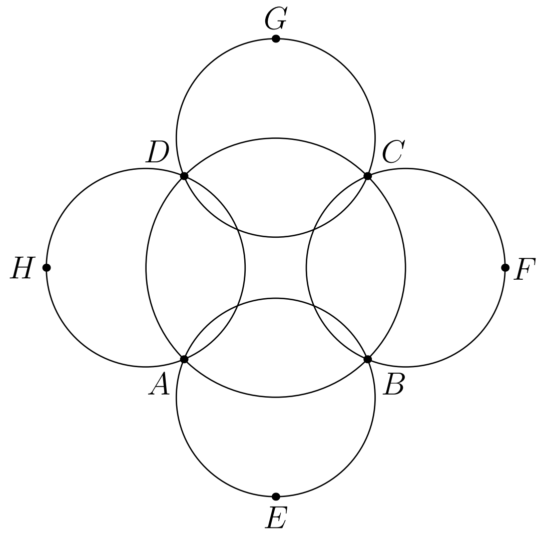
(Theresia Eisenkölbl) □

Problem 3. *The eight points A, B, …, G and H are placed on five circles as in the figure below. Each of these letters will be replaced with one of the numbers 1, 2, …, 7 and 8 such that the following two conditions hold:*

(i) Each of the eight numbers is used exactly once.

(ii) The sum of the numbers on each of the five circles is the same.

How many possibilities are there to replace the letters with numbers in this way?



(Walther Janous)

Answer. There are eight possibilities.

Solution. Let $a, b, \ldots, g$ and h be the numbers that replace the letters $A, B, \ldots, G$ and H. Since the sum on each circle has the value $a+b+c+d$, we immediately get $e = c+d$, $f = d+a$, $g = a+b$ and $h = b+c$. This implies $e+f+g+h = 2(a+b+c+d)$. On the other hand,

$$36 = 1+2+3+4+5+6+7+8 = (a+b+c+d)+(e+f+g+h) = 3(a+b+c+d),$$

and therefore $a+b+c+d = 12$.

Consequently, we get $e+g = (c+d)+(a+b) = 12$ and $f+h = (d+a)+(b+c) = 12$. Therefore, the pairwise distinct numbers e, f, g, h have to come from the set $\{4, 5, 7, 8\}$ where 8 and 4, as well as 7 and 5 lie on opposite points in the figure. There are eight possibilities to replace the letters E, F, G, H in this way with the numbers $4, 5, 7, 8$, because we can freely choose one of the four letters for 8 and then we have two choices for 7.

It remains to show that these choices of E, F, G, H determine the numbers in A, B, C, D uniquely. Without loss of generality, let $g = 8$ and $h = 7$, and therefore $e = 4$ and $f = 5$. From $e = 4 = c+d$ and $f = 5 = d+a$ we get that only b can take the value 6. The values $a = 2$, $c = 1$, $d = 3$ are now a direct consequence of the circle sums. Therefore, there are 8 possibilities.

(Stefan Leopoldseder) □

Problem 4. *Let p be a prime and let m and n be positive integers such that $p^2 + m^2 = n^2$.*
Prove that $m > p$.

(Karl Czakler)

Solution. We have $p^2 = n^2 - m^2 = (n-m)(n+m)$. Since p is a prime, the number p^2 has the divisors 1, p and p^2. Since the two factors $n-m$ and $n+m$ are distinct, they cannot be both equal to p. Furthermore, $n-m$ is smaller than $n+m$, therefore, $n-m = 1$, i. e. $n = m+1$.

We find

$$p^2 + m^2 = (m+1)^2 \iff p^2 = 2m+1.$$

This immediately implies that p is odd, therefore $p \geq 3$. We find $2m+1 = p^2 \geq 3p > 2p+1$, which gives $m > p$ as desired.

(Reinhard Razen) □