56^{th} Austrian Mathematical Olympiad

National Competition—Preliminary Round

3rd May 2025

1. Let a, b and c be three pairwise different nonnegative real numbers. Prove that

$$(a+b+c)\Big(\frac{a}{(b-c)^2}+\frac{b}{(c-a)^2}+\frac{c}{(a-b)^2}\Big)>4.$$

(Karl Czakler)

2. Let ABC be an acute triangle with $BC > AC$. Let S be the centroid of ABC and let F be the foot of the altitude from C to AB. The median CS intersects the circumcircle k of ABC a second time in P. It also intersects AB in M. The line SF intersects the circumcircle k in Q such that F lies between S and Q.

   Show that M, P, Q and F lie on a circle.

(Karl Czakler)

3. Consider the following game for a positive integer n: In the beginning, the numbers $1, 2, \dots, n$ are written on a blackboard. In each step, we select two numbers from the blackboard whose difference is still written on the blackboard. This difference is then erased from the blackboard.

   (For example, if the numbers left on the blackboard are 3, 6, 11 and 17, we can erase 3 as $6-3$ or 6 as $17-11$ or 11 as $17-6$.)

   Using such steps, for which n can we achieve that only one number remains on the blackboard?

(Michael Reitmeir)

4. Determine all integers z that can be represented in the form

$$z=\frac{a^2-b^2}{b},$$

   where a and b are positive integers.

(Walther Janous)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.