47^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 1)

April 30, 2016

Problem 1. *Determine the largest constant C such that*

$$(x_1 + x_2 + \cdots + x_6)^2 \geq C \cdot (x_1(x_2 + x_3) + x_2(x_3 + x_4) + \cdots + x_6(x_1 + x_2))$$

holds for all real numbers x_1, x_2, ..., x_6.

For this C, determine all x_1, x_2, ..., x_6 such that equality holds.

(Walther Janous)

Solution. We rewrite the right-hand side

$$x_1x_2 + x_1x_3 + x_2x_3 + x_2x_4 + x_3x_4 + x_3x_5 + x_4x_5 + x_4x_6 + x_5x_6 + x_1x_5 + x_1x_6 + x_2x_6$$

as

$$(x_1 + x_4)(x_2 + x_5) + (x_2 + x_5)(x_3 + x_6) + (x_3 + x_6)(x_1 + x_4).$$

Using the substitution $X = x_1 + x_4$, $Y = x_2 + x_5$ and $Z = x_3 + x_6$, the inequality reads

$$(X + Y + Z)^2 \geq C \cdot (XY + YZ + ZX),$$

where X, Y and Z are arbitrary real numbers.

For $X = Y = Z = 1$ we get $9 \geq 3C$, i.e., $C \leq 3$.

We now prove that

$$(X + Y + Z)^2 \geq 3(XY + YZ + ZX).$$

Expanding yields

$$X^2 + Y^2 + Z^2 \geq XY + YZ + ZX.$$

This is equivalent to

$$(X - Y)^2 + (Y - Z)^2 + (Z - X)^2 \geq 0$$

with equality for $X - Y = Y - Z = Z - X = 0$, i.e., $X = Y = Z$, thus $x_1 + x_4 = x_2 + x_5 = x_3 + x_6$.

(Walther Janous) □

Problem 2. *We are given an acute triangle ABC with $AB > AC$ and orthocenter H. The point E lies symmetric to C with respect to the altitude AH. Let F be the intersection of the lines EH and AC. Prove that the circumcenter of the triangle AEF lies on the line AB.*

(Karl Czakler)

Solution. See Figure 1.

Let θ be the angle between AF and the tangent t at A to the circumcircle of AEF. By the inscribed angle theorem, we have $\angle FEA = \theta$. Due to the reflection, we have $\angle ACH = \angle FEA = \theta$. Because of $\angle ACH = \theta$, the tangent t is parallel to CH and thus orthogonal to AB. Therefore, the circumcenter of the triangle AEF lies on AB.

Comment: This result also holds for obtuse triangles.

(Konstantin Mark, Clemens Heuberger) □

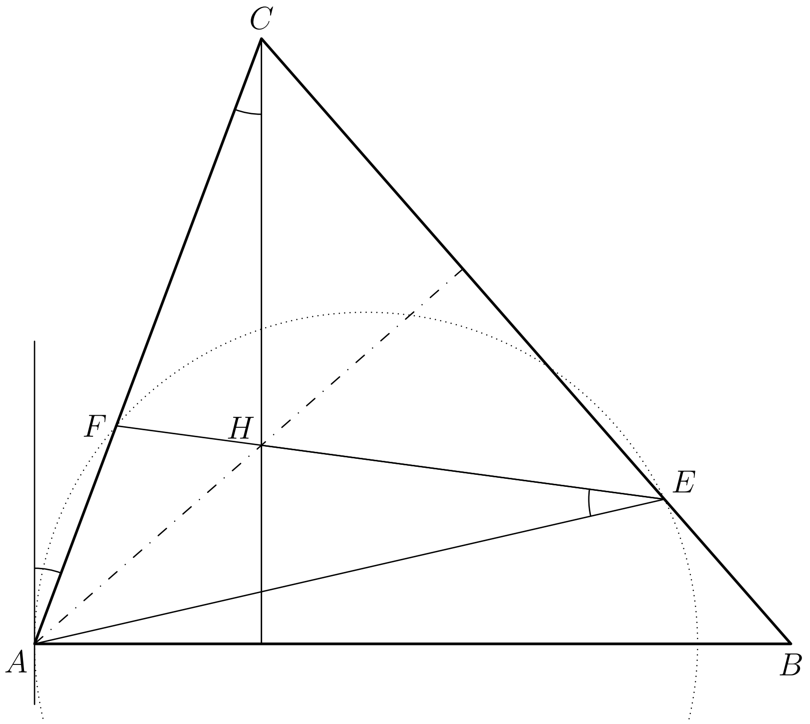

Figure 1: Problem 2

Problem 3. *Consider* 2016 *points arranged on a circle. We are allowed to jump ahead by* 2 *or* 3 *points in clockwise direction.*

What is the minimum number of jumps required to visit all points and return to the starting point?

(Gerd Baron)

Solution. Clearly, it takes at least 2016 jumps to visit all points. It is impossible to use only jumps of length 2 or only jumps of length 3 because this would confine us to a single residue class modulo 2 or 3, respectively.

If the problem could be solved with 2016 jumps, the total distance covered by these jumps would be strictly between $2 \cdot 2016$ and $3 \cdot 2016$ which makes a return to the original point impossible. Therefore, at least 2017 jumps are required.

This is indeed possible, for example with the following sequence of points on the circle.

$$0, 3, 6, \ldots, 2013, 2015, 2, 5, \ldots, 2012, 2014, 1, 4, \ldots, 2011, 2013, 0.$$

(Theresia Eisenkölbl) □

Problem 4. *Determine all composite positive integers n with the following property: If $1 = d_1 < d_2 < \ldots < d_k = n$ are all the positive divisors of n, then*

$$(d_2 - d_1) : (d_3 - d_2) : \cdots : (d_k - d_{k-1}) = 1 : 2 : \cdots : (k-1).$$

(Walther Janous)

Solution. Since n is a composite number, we have $k \geq 3$.

Let $d_2 = p$ be the smallest prime that divides n. We show by induction that

$$d_j = \frac{j(j-1)}{2}p - \frac{(j-2)(j+1)}{2}, \qquad j = 1, 2, \ldots, k.$$

This is clearly true for $j = 1$ and the induction step follows from $d_j - d_{j-1} = (j-1)(d_2 - d_1) = (j-1)(p-1)$ and $1 + 2 + 3 + \cdots + (j-1) = \frac{j(j-1)}{2}$.

If we apply this formula to $d_{k-1} = \frac{n}{p} = \frac{d_k}{d_2}$ and multiply by $2p$, we get

$$
\begin{aligned}
& (k-1)(k-2)p^2 - (k-3)kp = k(k-1)p - (k-2)(k+1) \\
\Leftrightarrow \quad & (k-1)(k-2)p^2 - 2(k-2)kp + (k-2)(k+1) = 0 \\
\Leftrightarrow \quad & (k-1)p^2 - 2kp + (k+1) = 0.
\end{aligned}
$$

The solutions of this quadratic equation are $p = 1$ and $p = \frac{k+1}{k-1} = 1 + \frac{2}{k-1}$. Since both options are at most 2, the only possibility is $p = 2$, $k = 3$ and $n = 4$. Since $n = 4$ has the required property, this is the only solution.

(Walther Janous) □