48^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 1)

30th April 2017

1. Determine all polynomials $P(x) \in \mathbb{R}[x]$ satisfying the following two conditions:
   (a) $P(2017) = 2016$ and
   (b) $(P(x) + 1)^2 = P(x^2 + 1)$ for all real numbers x.

   (Walther Janous)

2. Let $ABCDE$ be a regular pentagon with center M. A point $P \neq M$ is chosen on the line segment MD. The circumcircle of ABP intersects the line segment AE in A and Q and the line through P perpendicular to CD in P and R.

   Prove that AR and QR are of the same length.

   (Stephan Wagner)

3. Anna and Berta play a game in which they take turns in removing marbles from a table. Anna takes the first turn. When at the beginning of a turn there are $n \geq 1$ marbles on the table, then the player whose turn it is removes k marbles, where $k \geq 1$ either is an even number with $k \leq \frac{n}{2}$ or an odd number with $\frac{n}{2} \leq k \leq n$. A player wins the game if she removes the last marble from the table.

   Determine the smallest number $N \geq 100\,000$ such that Berta can enforce a victory if there are exactly N marbles on the table in the beginning.

   (Gerhard Woeginger)

4. Find all pairs (a, b) of non-negative integers such that

$$2017^a = b^6 - 32b + 1.$$

   (Walther Janous)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.