# $48^{\text{th}}$ Austrian Mathematical Olympiad

National Competition (Final Round, part 1)—Solutions

30th April 2017

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**Problem 1.** *Determine all polynomials $P(x) \in \mathbb{R}[x]$ satisfying the following two conditions:*

*(a)* $P(2017) = 2016$ *and*

*(b)* $(P(x)+1)^2 = P(x^2+1)$ *for all real numbers $x$.*

(Walther Janous)

*Solution.* Letting $Q(x) := P(x) + 1$ we get the two new conditions $Q(2017) = 2017$ and $Q(x^2+1) = Q(x)^2 + 1$, $x \in \mathbb{R}$.

We now define the sequence $\langle x_n \rangle_{n \geq 0}$ recursively by $x_0 = 2017$ and $x_{n+1} = x_n^2 + 1$, $n \geq 0$. A straightforward induction yields $Q(x_n) = x_n$, $n \geq 0$, because $Q(x_{n+1}) = Q(x_n^2+1) = Q(x_n)^2 + 1 = x_n^2 + 1 = x_{n+1}$.

Because of $x_0 < x_1 < x_2 < \cdots$ the two polynomials $Q(x)$ and $\mathrm{id}(x) = x$ coincide at infinitely many arguments $x$. Therefore, $Q(x) = x$ and thus the unique polynomial satisfying the two conditions of our problem is $P(x) = x - 1$.

*(Walther Janous)* □

**Problem 2.** *Let $ABCDE$ be a regular pentagon with center $M$. A point $P \neq M$ is chosen on the line segment $MD$. The circumcircle of $ABP$ intersects the line segment $AE$ in $A$ and $Q$ and the line through $P$ perpendicular to $CD$ in $P$ and $R$.*

*Prove that $AR$ and $QR$ are of the same length.*

(Stephan Wagner)

*Solution.* Let $S$ denote the common point of $RP$ and $AE$, see Figure 1. Since we are given a regular pentagon, the angles in triangle $ABE$ are well known as $\angle BAE = 108°$ and $\angle ABE = \angle AEB = 36°$. Since $BE$ and $CD$ are parallel, $RP$ is perpendicular to $BE$, and we therefore have $\angle ASP = 126°$ and $\angle QSP = 54° = \angle ASR$. From this,

$$\angle SPA = 54° - \angle SAP = \angle PAB - 54° = \angle PBA - 54° = 126° - \angle AQP = 126° - \angle SQP = \angle SPQ$$

follows, since $ABPQ$ is inscribed. We therefore see that $SP$ (or $RP$) bisects the angle $\angle APQ$, which implies that $AR$ and $QR$ must be of equal length, as claimed.

*(Stephan Wagner)* □

**Problem 3.** *Anna and Berta play a game in which they take turns in removing marbles from a table. Anna takes the first turn. When at the beginning of a turn there are $n \geq 1$ marbles on the table, then the player whose turn it is removes $k$ marbles, where $k \geq 1$ either is an even number with $k \leq \frac{n}{2}$ or an odd number with $\frac{n}{2} \leq k \leq n$. A player wins the game if she removes the last marble from the table.*

*Determine the smallest number $N \geq 100\,000$ such that Berta can enforce a victory if there are exactly $N$ marbles on the table in the beginning.*

(Gerhard Woeginger)

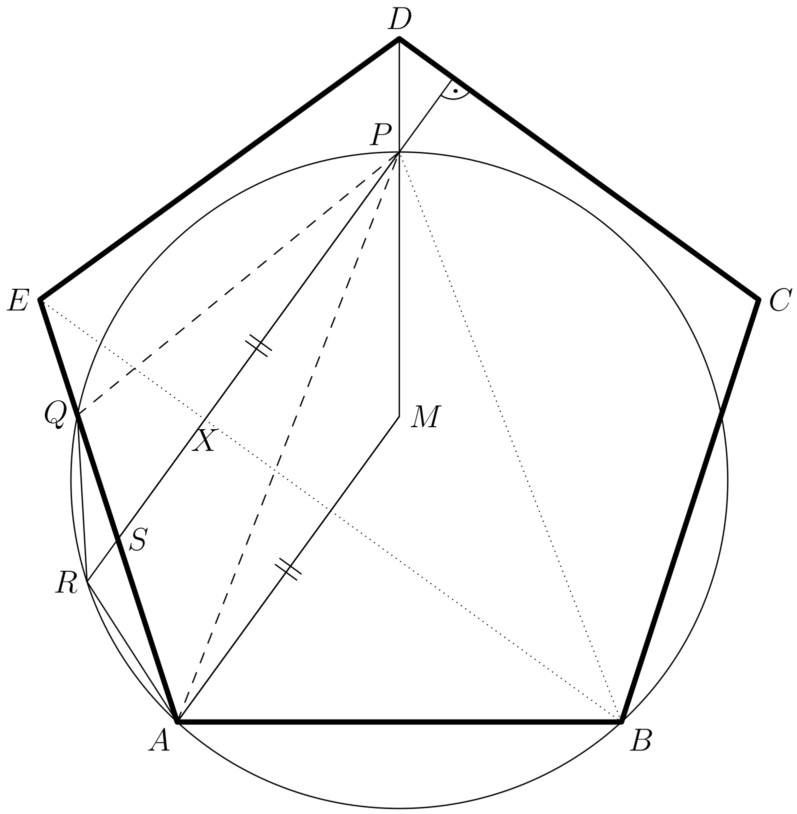


Figure 1: Problem 2

*Solution.* We claim that the losing situations are those with exactly $n = 2^a - 2$ marbles left on the table for all integers $a \geq 2$. All other situations are winning situations.

Proof: By induction for $n \geq 1$. For $n = 1$ the player wins by taking the single remaining marble. For $n = 2$ the only possible move is to take $k = 1$ marbles, and then the opponent wins in the next move.

Induction step from $n - 1$ to $n$ for $n \geq 3$:

1. If $n$ is odd, then the player takes all $n$ marbles and wins.

2. If $n$ is even but not of the form $2^a - 2$, then $n$ lies between two other numbers of that form, so there exists a unique $b$ with $2^b - 2 < n < 2^{b+1} - 2$. Because of $n \geq 3$ it holds that $b \geq 2$. Therefore all three numbers in this chain of inequalities are even, and therefore we can conclude that $2^b \leq n \leq 2^{b+1} - 4$. From the induction hypothesis we know that $2^b - 2$ is a losing situation, and by taking
$$k = n - (2^b - 2) = n - \frac{2^{b+1} - 4}{2} \leq n - \frac{n}{2} = \frac{n}{2}$$
marbles we leave it to the opponent.

3. If $n$ is even and of the form $n = 2^a - 2$, then the player cannot leave a losing situation with $2^b - 2$ marbles to the opponent (where $b < a$ holds because at least one marble must be removed, and $b \geq 2$ holds because after a legal move starting from an even $n$, at least one marble remains). In order to do so, the player would have to remove $k = (2^a - 2) - (2^b - 2) = 2^a - 2^b$ marbles. But because of $b \geq 2$ we know that $k$ is even and strictly greater than $\frac{n}{2}$ because of $2^a - 2^b \geq 2^a - 2^{a-1} = 2^{a-1} > 2^{a-1} - 1 = \frac{2^a - 2}{2} = \frac{n}{2}$; impossible.

Solution: Berta can enforce a victory if and only if $N$ is of the form $2^a - 2$. The smallest number $N \geq 100\,000$ of this form is $N = 2^{17} - 2 = 131\,070$.

*(Gerhard Woeginger)* □

**Problem 4.** *Find all pairs $(a, b)$ of non-negative integers such that*

$$2017^a = b^6 - 32b + 1.$$

(Walther Janous)

*Solution.* Answer: The two solutions are $(0, 0)$ and $(0, 2)$.

Since $2017^a$ is always odd, $b$ must be even, so $b = 2c$, $c$ integer. Therefore, $2017^a = 64(c^6 - c) + 1$ and thus $2017^a \equiv 1 \pmod{64}$. But we find $2017 \equiv 33 \pmod{64}$ and $2017^2 \equiv (1+32)^2 = 1 + 2 \cdot 32 + 32^2 \equiv 1 \pmod{64}$, so that the powers of 2017 modulo 64 alternate between 1 and 33. Therefore, $a$ is even and $2017^a$ is a perfect square. We denote the polynomial on the right-hand side of the given equation by $r(b) = b^6 - 32b + 1$ and show that it lies between two consecutive squares for $b > 4$:

Let $b > 4$. We have $r(b) < b^6 = (b^3)^2$ for $b > 0$. On the other hand, $r(b) > (b^3 - 1)^2$ because $b^6 - 32b + 1 > b^6 - 2b^3 + 1 \Leftrightarrow b > 4$. Since the square $2017^a$ is now between two consecutive squares, there are no solutions in this case.

Since $b$ is even, it remains to check $b = 4$, $b = 2$ and $b = 0$.

For $b = 4$, we regard the equation modulo 3 and get $1 \equiv 1 - 2 + 1 = 0$, therefore, there is no solution in this case.

For $b = 2$, we get $2017^a = 2^6 - 2^6 + 1$, so we get the solution $(a, b) = (0, 2)$.

For $b = 0$, we get $2017^a = 1$, so we get the solution $(a, b) = (0, 0)$.

Therefore, $(0, 0)$ and $(0, 2)$ are the only solutions.

*(Walther Janous)* □