

49th Austrian Mathematical Olympiad
 Regional Competition (Qualifying Round)—Solutions
 5th April 2018

Problem 1. Let a and b be nonnegative real numbers satisfying $a + b < 2$. Prove the inequality

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} \leq \frac{2}{1+ab}$$

and determine all a and b yielding equality.

(Gottfried Perz)

Solution. (Karl Czakler) We have

$$\frac{1}{1+a^2} + \frac{1}{1+b^2} = \frac{2+a^2+b^2}{(1+a^2)(1+b^2)} = 1 + \frac{1-a^2b^2}{(1+a^2)(1+b^2)}.$$

Via the Cauchy-Schwarz inequality we get $(1+a^2)(1+b^2) \geq (1+ab)^2$. Furthermore, the arithmetic-geometric means inequality yields $ab \leq \left(\frac{a+b}{2}\right)^2 < 1$. Therefore, the inequality

$$1 + \frac{1-a^2b^2}{(1+a^2)(1+b^2)} \leq 1 + \frac{1-a^2b^2}{(1+ab)^2} = \frac{2}{1+ab}$$

follows and we are done. Finally, equality occurs if and only if $0 \leq a = b < 1$. □

Problem 2. Let k be a circle with radius r and AB a chord of k such that $AB > r$. Furthermore, let S be the point on the chord AB satisfying $AS = r$. The perpendicular bisector of BS intersects k in the points C and D . The line through D and S intersects k for a second time in point E . Show that the triangle CSE is equilateral.

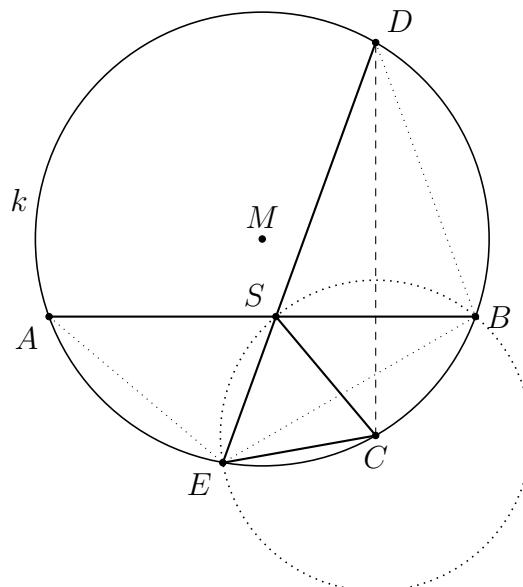
(Stefan Leopoldseder)

Solution. (Stefan Leopoldseder) In the first part we prove $CS = CE$, afterwards we will prove $\angle SCE = 60^\circ$.

Since C and D lie on the symmedian of BS , $CBDS$ is a kite with

$$CS = CB \quad \text{and} \quad \angle CDB = \angle SDC = \angle EDC.$$

The line segments CB and CE therefore have equal length (equal inscribed angles at D). Hence, we have shown that $CS = CB = CE$. Furthermore there exists a circle k_1 with C containing E , S and B .



The triangle ESA is isosceles with base ES , because it is similar to the isosceles triangle BSD with base BS ($\angle SEA = \angle DEA = \angle DBA = \angle DBS$ and $\angle EAS = \angle EAB = \angle EDB = \angle SDB$ follow from the inscribed angle theorem in k). We therefore have $AE = AS = r$, that is, the length of the chord AE is equal to the radius r of the circle k . The central angle for the chord AE is therefore 60° , the inscribed angle $\angle ABE$ is $60/2 = 30^\circ$. But now $\angle SBE = \angle ABE = 30^\circ$ is an inscribed angle in k_1 for the chord SE , hence the central angle is $\angle SCE = 2 \cdot 30 = 60^\circ$. \square

Problem 3. Let $n \geq 3$ be a natural number.

Determine the number a_n of all subsets of $\{1, 2, \dots, n\}$ consisting of three elements such that one of them is the arithmetic mean of the other two.

(Walther Janous)

Solution. (Gerd Baron) Let $\{a, b, c\}$ be a subset of $M = \{1, 2, \dots, n\}$ such that $a < b < c$. Then $b = (a + c)/2$ is an element of M if and only if a and c have the same parity.

- Let $n = 2k$ be even. Then the sets $\{a, c\}$ are the $k(k-1)/2$ subsets of M consisting of the k even as well of the k odd numbers in M . Therefore, the desired number of subsets $\{a, b, c\}$ of M equals $a_{2k} = k(k-1)$.
- If $n = 2k + 1$ is odd, we similarly get $k(k-1)/2$ subsets $\{a, c\}$ consisting of two even numbers a and c as well as $(k+1)k/2$ such subsets with odd numbers. Thus, $a_{2k+1} = k^2$.

The two results can be summarized as $a_n = \lfloor \frac{n}{2} \rfloor \cdot \lfloor \frac{n-1}{2} \rfloor$. \square

Problem 4. Let $d(n)$ be the number of all positive divisors of a natural number $n \geq 2$.

Determine all natural numbers $n \geq 3$ such that

$$d(n-1) + d(n) + d(n+1) \leq 8.$$

(Richard Henner)

Solution. (Gerhard Kirchner) For even numbers $k \geq 6$ we have $d(k) \geq 4$, since $1, 2, \frac{k}{2}, k$ are four different divisors. It is clear that $n = 3$ is a solution, whereas $n = 5$ is not. For odd numbers $n \geq 7$ we have

$$d(n-1) + d(n) + d(n+1) \geq 4 + d(n) + 4 > 8.$$

From now on, let n be even. If the number $k \geq 6$ is divisible by 3, we have $d(k) \geq 3$, since $1, 3, k$ are three different divisors. We check that $n = 4$ and $n = 6$ satisfy the condition. If $n \geq 8$ and $n-1$ is divisible by 3, then

$$d(n-1) + d(n) + d(n+1) \geq 3 + 4 + d(n+1) > 8.$$

If, on the other hand, $n \geq 8$ and $n+1$ is divisible by 3, then we have

$$d(n-1) + d(n) + d(n+1) \geq d(n-1) + 4 + 3 > 8.$$

Since among the three successive integers $n-1, n, n+1$ one has to be divisible by 3, the only remaining case is that n is divisible by 6. In this case n has the six different divisors $1, 2, 3, \frac{n}{3}, \frac{n}{2}, n$, i. e. $d(n) \geq 6$ for $n \geq 12$. Thus, we get

$$d(n-1) + d(n) + d(n+1) \geq d(n-1) + 6 + d(n+1) > 8.$$

Hence, there are no further solutions. \square