44^{th} Austrian Mathematical Olympiad

Beginners' Competition

June 13^{th}, 2013

Problem 1. *Find all integers $n > 1$ such that the sum of n and its second-largest divisor is 2013.*

R. Henner, Vienna

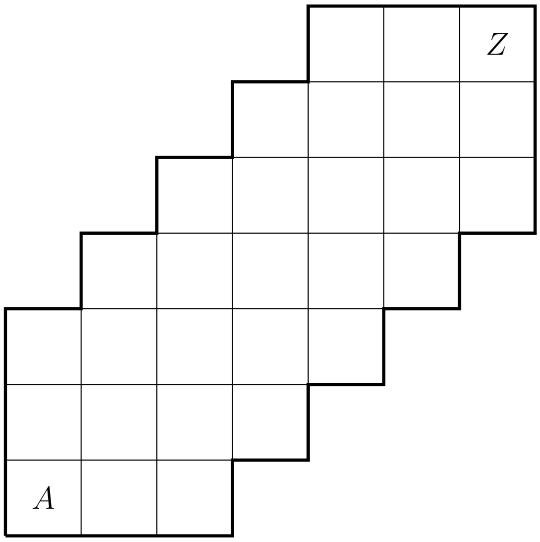
Solution. The second-largest divisor of n is of the form $\frac{n}{p}$ where p is the smallest prime that divides n.

The given condition gives $2013 = n + \frac{n}{p} = \frac{n}{p}(p+1)$. Therefore, $p+1$ is a divisor of 2013 and thus odd. So, p is 2, the only even prime.

The equation now becomes $2013 = \frac{n}{2} \cdot 3$ which gives the unique solution $n = 1342$.

(T. Eisenkölbl) □

Problem 2. *Find the number of paths from square A to square Z in the figure below where a path consists of steps from a square to its upper or right neighbouring square.*



W. Janous, WRG Ursulinen, Innsbruck

Solution. For every square in the figure, we will count the number of ways from square A to this square and will write this number inside the square.

For A itself, this number is 1, since there is only one way to stay in A.

If a square has a left and a lower neighbour square, the number of paths to this square is the sum of the two numbers in these two neighbours.

a	$a+b$
	b

If a square has only one of these two neighbours, then all paths have to pass through this neighbour, so the number of ways from A is the same for the square and its neighbour.

a	a

a
a

Therefore, we can recursively fill the squares with the numbers of paths from A. We get the following result:

				81	243	486
			27	81	162	243
		9	27	54	81	81
	3	9	18	27	27	
1	3	6	9	9		
1	2	3	3			
1	1	1				

The final answer for the number of ways from A to Z is therefore 486.

(T. Eisenkölbl) $\square$

Problem 3. *Let a and b be real numbers with $0 \leq a,\ b \leq 1$.*

Prove that

$$\frac{a}{b+1} + \frac{b}{a+1} \leq 1$$

and find the cases of equality.

K. Czakler, Vienna

Solution. We clear denominators to get

$$\begin{aligned}
& & a(a+1)+b(b+1) &\le (a+1)(b+1),\\
&\Leftrightarrow & a^2+a+b^2+b &\le ab+a+b+1,\\
&\Leftrightarrow & a^2-a+b^2-b &\le ab-a-b+1,\\
&\Leftrightarrow & a(a-1)+b(b-1) &\le (a-1)(b-1),\\
&\Leftrightarrow & (1-a)(1-b)+a(1-a)+b(1-b) &\ge 0.
\end{aligned}$$

The three terms on the left-hand side of the last inequality are clearly all positive or zero for $0 \le a, b \le 1$.

For equality to hold, all three terms have to be zero, that is, $a = 1$ or $b = 1$ and $a, b \in \{0, 1\}$.

This gives the three pairs $(a, b) = (1, 0)$, $(a, b) = (0, 1)$ and $(a, b) = (1, 1)$.

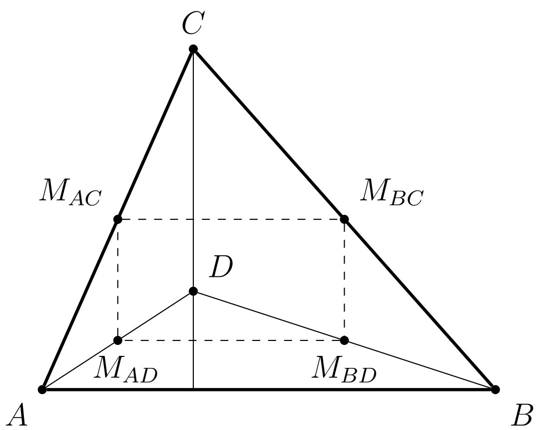
(T. Eisenkölbl) $\square$

Problem 4. *Let ABC be an acute triangle and D be a point on the altitude through C.*

Prove that the mid-points of the line segments AD, BD, BC and AC form a rectangle.

G. Anegg, Innsbruck

Solution. The problem is represented in the following figure:



We denote with M_{XY} the mid-point of the line segment XY.

Using the intercept theorem, we deduce that

- $M_{AD}M_{BD}$ is parallel to AB.
- $M_{AC}M_{BC}$ is parallel to AB.
- $M_{AC}M_{AD}$ is parallel to CD.
- $M_{BC}M_{BD}$ is parallel to CD.

Therefore, $M_{AD}M_{BD}$ is parallel to $M_{AC}M_{BC}$ and $M_{AC}M_{AD}$ is parallel to $M_{BC}M_{BD}$. Furthermore, $M_{AC}M_{BC}$ is orthogonal to $M_{AC}M_{AD}$, since CD is orthogonal to AB. Therefore, the four mid-points form a rectangle.

(G. Anegg) $\square$