Czech-Polish-Slovak Match

IST Austria, 23 – 26 June 2019

(Second day – 25 June 2019)

4. Let α be a given real number. Determine all pairs (f, g) of functions $f, g\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$xf(x+y) + \alpha \cdot yf(x-y) = g(x) + g(y)$$

for all $x, y \in \mathbb{R}$. (*Walther Janous, Austria*)

Solution. Depending on α, the solutions are given by:

- If $\alpha = 1$, then $f(x) = C$ and $g(x) = Cx$ for $x \in \mathbb{R}$ and C an arbitrary real constant.
- If $\alpha = -1$, then $f(x) = Cx$ and $g(x) = Cx^2$ for $x \in \mathbb{R}$ and C an arbitrary real constant.
- Else, $f(x) = g(x) = 0$ for $x \in \mathbb{R}$.

Letting $x = y = 0$, we obtain $2g(0) = 0$, thus $g(0) = 0$. Letting $y = 0$, we obtain $xf(x) = g(x)$ for all $x \in \mathbb{R}$. Thus, the equation can be rewritten as

$$xf(x+y) + \alpha yf(x-y) = xf(x) + yf(y). \tag{1}$$

Letting $x = 0$ in (1), we obtain $\alpha yf(-y) = yf(y)$. This yields

$$\forall x \neq 0\colon f(-x) = \alpha f(x). \tag{2}$$

If $f(x) = 0$ for all $x \neq 0$, we let $y = -x \neq 0$ in (1) and obtain $xf(0) = 0$, therefore f is the zero function, which always solves the equation.

Assume now that there exists $r \neq 0$ with $f(r) \neq 0$. Then it follows from (2) that $f(r) = \alpha f(-r) = \alpha^2 f(r)$, thus $\alpha^2 = 1$ and hence $\alpha \in \{\pm 1\}$.

The right-hand side of (1) is symmetric in x and y. By switching x and y, we thus obtain the equation

$$xf(x+y) + \alpha yf(x-y) = yf(x+y) + \alpha xf(y-x).$$

For $r \in \mathbb{R}$ we let $x = (r+1)/2$ and $y = (r-1)/2$, which yields

$$f(r) = \alpha\frac{r+1}{2}f(-1) - \alpha\frac{r-1}{2}f(1).$$

By (2), we obtain

$$f(r) = \frac{\alpha f(1)}{2}\Big(\alpha(r+1) - (r-1)\Big).$$

In the case $\alpha = 1$ this means $f(r) = f(1)$ for all $r \in \mathbb{R}$. In the case $\alpha = -1$ this means $f(r) = rf(1)$ for all $r \in \mathbb{R}$. Both functions solve the equation, as can be checked easily.

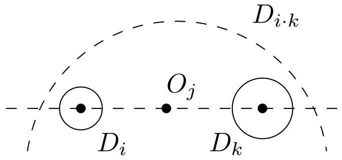
5. Determine whether there exist 100 disks $D_2, D_3, \ldots, D_{101}$ in the plane such that the following conditions hold for all pairs (a, b) of indices satisfying $2 \le a < b \le 101$:

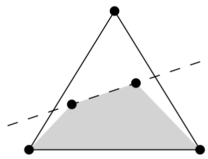
1. If $a \mid b$ then D_a is contained in D_b.
2. If $\mathrm{GCD}(a, b) = 1$ then D_a and D_b are disjoint.

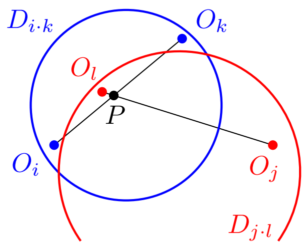
(A disk $D(O, r)$ is a set of points in the plane whose distance to a given point O is at most a given positive real number r.) (*Josef Greilhuber & Josef Tkadlec, Austria*)

Solution. Such disks do not exist. Suppose otherwise and denote by O_i the center of the disk D_i. Consider the set $S = \{O_2, O_3, O_5, O_7, O_{11}\}$ of centers of five disks with pairwise coprime indices. We distinguish two cases:

(i) Some three points from S lie on a single line: Suppose the three collinear points are O_i, O_j, O_k in this order. Then $i \cdot k \le 7 \cdot 11 \le 101$, hence the disk $D_{i \cdot k}$ is defined. By 1., it contains both D_i and D_k, thus it contains O_i and O_k and by convexity it also contains O_j. Therefore, disks D_j, $D_{i \cdot k}$ intersect, a contradiction with 2.



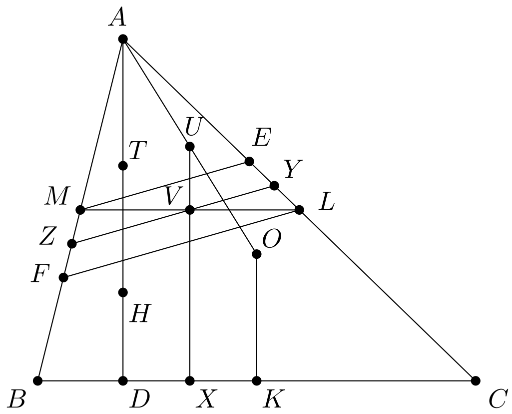




(ii) No three points from S lie on a single line: Then there exist four points from S that form a convex quadrilateral. (Indeed, either the convex hull of S contains at least four points, or it is a triangle. In the latter case, the line passing through the two interior points intersects two sides of the triangle and the two interior points form a convex quadrilateral with the endpoints of the side that is not intersected.) Suppose the four vertices of the convex quadrilateral are O_i, O_j, O_k, O_l in this order. Then, as before, both $i \cdot k$ and $j \cdot l$ are at most $7 \cdot 11 \le 101$ hence the disks $D_{i \cdot k}$ and $D_{j \cdot l}$ are defined. By 1. and by convexity, they both contain the intersection P of diagonals of $O_i O_j O_k O_l$, which is a contradiction with 2.

6. Let ABC be an acute triangle with $AB < AC$ and $\angle BAC = 60^\circ$. Denote its altitudes by AD, BE, CF and its orthocenter by H. Let K, L, M be the midpoints of sides BC, CA, AB, respectively. Prove that the midpoints of segments AH, DK, EL, FM lie on a single circle. (*Dominik Burek, Poland*)

Solution. Denote the midpoints of AH, DK, EL, FM by T, X, Y, Z, respectively. Furthermore, let O be the circumcenter of triangle ABC and U the midpoint of AO



(that is, the circumcenter of triangle AML). We will show that U lies on the circle too.

First, we show that $TUYZ$ is cyclic. In fact, we show that is is an isosceles trapezoid whose line of symmetry is the angle bisector of $\angle BAC$: Since $\angle BAC = 60°$, we have $AE = \frac{1}{2}AB = AM$, thus $\triangle AME$ is equilateral and, likewise, $\triangle AFL$ is equilateral. Since Y and Z are the midpoints of lateral sides EL, MF of a trapezoid $ELFM$, triangle AYZ is also equilateral and the perpendicular bisector of YZ is the angle bisector of $\angle BAC$. Regarding TU, since lines AT and AU are isogonal in $\angle BAC$ and $AF = AL$, the right triangles AFH and ALO are congruent. Thus the perpendicular bisector of TU is the angle bisector of $\angle BAC$ as well.

Second, we show that $UYXZ$ is cyclic: Let V be the center of parallelogram $AMKL$. Since V is the midpoint of ML, it lies on the midline YZ of trapezoid $MELF$. Since it is the midpoint of AK, it also lies on the midline UX of trapezoid $AOKD$. Thus, it remains to check that $VY \cdot VZ = VU \cdot VX$, which is straightforward. For the left-hand side, we have $VY = \frac{1}{2}ME = \frac{1}{4}AB$ and $VZ = \frac{1}{2}LF = \frac{1}{2}AF$. For the right-hand side, we have $VU = \frac{1}{2}OK = \frac{1}{4}AH$ and $VX = \frac{1}{2}AD$. Plugging this in, we need $AB \cdot AF = AH \cdot AD$ which follows from $BFHD$ being cyclic.

Since both $TUYZ$ and $UYXZ$ are cyclic, so is $TYXZ$.