53^{rd} Austrian Mathematical Olympiad

National Competition—Preliminary Round—Solutions

30th April 2022

Problem 1. *Prove that for all real positive numbers x, y and z the double inequality*

$$0 < \frac{1}{x+y+z+1} - \frac{1}{(x+1)(y+1)(z+1)} \leq \frac{1}{8}$$

holds.

For which values does equality hold in the right-hand inequality?

(Walther Janous)

Answer. Equality holds for $x = y = z = 1$.

Solution.

- The left-hand side is immediate from

$$(x+1)(y+1)(z+1) = x+y+z+1+xy+yz+zx+xyz > x+y+z+1.$$

- For the right-hand side, we apply the AM-GM-inequality to the second denominator, i.e.

$$(x+1)(y+1)(z+1) \leq \left(\frac{x+y+z+3}{3}\right)^3$$

and obtain with $s = (x+y+z)/3$ that

$$\frac{1}{x+y+z+1} - \frac{1}{(x+1)(y+1)(z+1)} \leq \frac{1}{3s+1} - \frac{1}{(s+1)^3}$$

with equality for $x = y = z$.

It remains to show for $s > 0$ that

$$\begin{aligned} & \frac{1}{3s+1} - \frac{1}{(s+1)^3} \leq \frac{1}{8} \\ \iff\; & 8((s+1)^3 - (3s+1)) \leq (3s+1)(s+1)^3 \\ \iff\; & 3s^4 + 2s^3 - 12s^2 + 6s + 1 \geq 0 \\ \iff\; & (s-1)^2(3s^2 + 8s + 1) \geq 0, \end{aligned}$$

which is clearly true with equality for $s = 1$.

Therefore, the right-hand side inequality is true with equality for $x = y = z$ and $s = 1$ which means $x = y = z = 1$.

(Walther Janous) □

Problem 2. *Points A, B, C and D lie on a circle in this order. Let O be the circle's center. Suppose AC and BD are orthogonal. Let F be the foot of the altitude from O to AB.*

Prove that $CD = 2 \cdot OF$.

(Karl Czakler)

Solution. Let R be the radius of the given circle. By the sine law, we get

$$CD = 2R \sin \angle CAD,$$

and we also have

$$OF = R \sin \angle FAO$$

in the right triangle FAO.

On the other hand, $\angle FOA = \angle BDA$ by the inscribed angle theorem, and, using the given right angles, also

$$\angle FAO = \angle CAD,$$

which finishes the proof.

(Theresia Eisenkölbl) □

Problem 3. *At each integer on the number line from 0 through 2022, a person is standing at the start of a process.*

In each move, two of these people, standing at least two units apart, are chosen. Each of these walks one unit closer to the other.

If no further move is possible, the process ends.

Prove that this process must terminate after a finite number of moves and determine all possible final configurations where the persons can stand. (The configurations only take into account how many persons stand at each number.)

(Birgit Vera Schmidt)

Answer. In the final configuration, all people will be standing on 1011.

Solution. We use the sum of all pairwise distances $\sum_{i,j} |p_i - p_j|$, where $p_1, p_2, \ldots$ are the positions of the persons, and we will show that it gets smaller in every step of the process.

If two people in positions a and b with $a < b$ go towards each other than the distances between other people remain unchanged. For each person in a position $\geq b$, the distance to one of the two persons will increase by one and the distance to the other one will decrease by one, so that the sum of distances does not change. The same is true for each person in a position $\leq a$.

The distance between the two chosen persons will get smaller and the distances between a chosen person and the people between them will also get smaller. Therefore, the sum of distances is strictly decreasing, but also an integer greater or equal to zero, so the process must end.

We also see that the average of all positions is an invariant and therefore, will end at the same value 1011. We observe that at the end at most two neighboring positions can be used and the average will not be an integer if there are actually two that are used. Therefore, everyone must be in the same positions, namely the average 1011.

(Theresia Eisenkölbl) □

Problem 4. *Determine all triples (p, q, r) of prime numbers such that $4q - 1$ is a prime number, too, and*

$$\frac{p+q}{p+r} = r - p.$$

(Walther Janous)

Answer. There is a unique triple of prime numbers solving the problem: $(p, q, r) = (2, 3, 3)$.

Solution. The given equation is equivalent to

$$\begin{aligned} & q = r^2 - p^2 - p \\ \iff & 4q - 1 = 4r^2 - (4p^2 + 4p + 1) \\ \iff & 4q - 1 = (2r - 2p - 1)(2r + 2p + 1). \end{aligned}$$

Since $4q - 1$ is prime and $2r + 2p + 1 > 1$ ist, the first factor must be $2r - 2p - 1 = 1$, i.e.

$$r = p + 1.$$

The only primes with distance 1 are $p = 2$ and $r = 3$. For these values, we get

$$(2r - 2p - 1)(2r + 2p + 1) = 11,$$

which is indeed a prime and because of $4q - 1 = 11$, we have $q = 3$.

Therefore, the only solution is $(p, q, r) = (2, 3, 3)$.

(Walther Janous) □