

46th Austrian Mathematical Olympiad
National Competition (Final Round, part 2, first day)
May 20, 2015

1. Let $f: \mathbb{Z}_{>0} \rightarrow \mathbb{Z}$ be a function with the following properties:

- (i) $f(1) = 0$,
- (ii) $f(p) = 1$ for all prime numbers p ,
- (iii) $f(xy) = yf(x) + xf(y)$ for all x, y in $\mathbb{Z}_{>0}$.

Determine the smallest integer $n \geq 2015$ that satisfies $f(n) = n$.

(Gerhard J. Woeginger)

2. We are given a triangle ABC . Let M be the mid-point of its side AB .

Let P be an interior point of the triangle. We let Q denote the point symmetric to P with respect to M .

Furthermore, let D and E be the common points of AP and BP with sides BC and AC , respectively.

Prove that points A, B, D and E lie on a common circle if and only if $\angle ACP = \angle QCB$ holds.

(Karl Czakler)

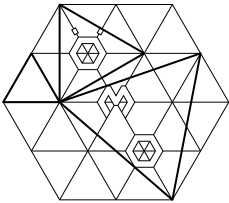
3. We consider the following operation applied to a positive integer: The integer is represented in an arbitrary base $b \geq 2$, in which it has exactly two digits and in which both digits are different from 0. Then the two digits are swapped and the result in base b is the new number.

Is it possible to transform every number > 10 to a number ≤ 10 with a series of such operations?

(Theresia Eisenkölbl)

Working time: $4\frac{1}{2}$ hours.

Each problem is worth 8 points.



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4. Let x, y, z be positive real numbers with $x + y + z \geq 3$. Prove that

$$\frac{1}{x + y + z^2} + \frac{1}{y + z + x^2} + \frac{1}{z + x + y^2} \leq 1.$$

When does equality hold?

(Karl Czakler)

5. Let I be the incenter of triangle ABC and let k be a circle through the points A and B . This circle intersects

- the line AI in points A and P ,
- the line BI in points B and Q ,
- the line AC in points A and R and
- the line BC in points B and S ,

with none of the points A, B, P, Q, R and S coinciding and such that R and S are interior points of the line segments AC and BC , respectively.

Prove that the lines PS, QR and CI meet in a single point.

(Stephan Wagner)

6. Max has 2015 jars labelled with the numbers 1 to 2015 and an unlimited supply of coins. Consider the following starting configurations:

- (a) All jars are empty.
- (b) Jar 1 contains 1 coin, jar 2 contains 2 coins, and so on, up to jar 2015 which contains 2015 coins.
- (c) Jar 1 contains 2015 coins, jar 2 contains 2014 coins, and so on, up to jar 2015 which contains 1 coin.

Now Max selects in each step a number n from 1 to 2015 and adds n coins to each jar *except to the jar n* .

Determine for each starting configurations in (a), (b), (c), if Max can use a finite, strictly positive number of steps to obtain an equal number of coins in each jar.

(Birgit Vera Schmidt)

Working time: $4\frac{1}{2}$ hours.

Each problem is worth 8 points.