46^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 2, first day)

May 20, 2015

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1. Let $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}$ be a function with the following properties:

   (i) $f(1) = 0$,

   (ii) $f(p) = 1$ for all prime numbers p,

   (iii) $f(xy) = yf(x) + xf(y)$ for all x, y in $\mathbb{Z}_{>0}$.

   Determine the smallest integer $n \geq 2015$ that satisfies $f(n) = n$.

   (Gerhard J. Woeginger)

2. We are given a triangle ABC. Let M be the mid-point of its side AB.

   Let P be an interior point of the triangle. We let Q denote the point symmetric to P with respect to M.

   Furthermore, let D and E be the common points of AP and BP with sides BC and AC, respectively.

   Prove that points A, B, D and E lie on a common circle if and only if $\angle ACP = \angle QCB$ holds.

   (Karl Czakler)

3. We consider the following operation applied to a positive integer: The integer is represented in an arbitrary base $b \geq 2$, in which it has exactly two digits and in which both digits are different from 0. Then the two digits are swapped and the result in base b is the new number.

   Is it possible to transform every number > 10 to a number ≤ 10 with a series of such operations?

   (Theresia Eisenkölbl)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

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4. Let x, y, z be positive real numbers with $x+y+z \geq 3$. Prove that

$$\frac{1}{x+y+z^2} + \frac{1}{y+z+x^2} + \frac{1}{z+x+y^2} \leq 1.$$

When does equality hold?

(Karl Czakler)

5. Let I be the incenter of triangle ABC and let k be a circle through the points A and B. This circle intersects

- the line AI in points A and P,
- the line BI in points B and Q,
- the line AC in points A and R and
- the line BC in points B and S,

with none of the points A, B, P, Q, R und S coinciding and such that R and S are interior points of the line segments AC and BC, respectively.

Prove that the lines PS, QR and CI meet in a single point.

(Stephan Wagner)

6. Max has 2015 jars labelled with the numbers 1 to 2015 and an unlimited supply of coins.

Consider the following starting configurations:

(a) All jars are empty.

(b) Jar 1 contains 1 coin, jar 2 contains 2 coins, and so on, up to jar 2015 which contains 2015 coins.

(c) Jar 1 contains 2015 coins, jar 2 contains 2014 coins, and so on, up to jar 2015 which contains 1 coin.

Now Max selects in each step a number n from 1 to 2015 and adds n coins to each jar *except to the jar n.*

Determine for each starting configurations in (a), (b), (c), if Max can use a finite, strictly positive number of steps to obtain an equal number of coins in each jar.

(Birgit Vera Schmidt)

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