48^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 2, first day)

24th May 2017

1. Let α be a fixed real number.

   Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

   $$f(f(x+y)f(x-y)) = x^2 + \alpha y f(y)$$

   for all x, $y \in \mathbb{R}$.

   (Walther Janous)

2. A necklace contains 2016 pearls, each of which has one of the colours black, green or blue. In each step we replace simultaneously each pearl with a new pearl, where the colour of the new pearl is determined as follows: If the two original neighbours were of the same colour, the new pearl has their colour. If the neighbours had two different colours, the new pearl has the third colour.

   (a) Is there such a necklace that can be transformed with such steps to a necklace of blue pearls if half of the pearls were black and half of the pearls were green at the start?

   (b) Is there such a necklace that can be transformed with such steps to a necklace of blue pearls if thousand of the pearls were black at the start and the rest green?

   (c) Is it possible to transform a necklace that contains exactly two adjacent black pearls and 2014 blue pearls to a necklace that contains one green pearl and 2015 blue pearls?

   (Theresia Eisenkölbl)

3. Let $(a_n)_{n\geq 0}$ be the sequence of rational numbers with $a_0 = 2016$ and

   $$a_{n+1} = a_n + \frac{2}{a_n}$$

   for all $n \geq 0$.

   Show that the sequence does not contain a square of a rational number.

   (Theresia Eisenkölbl)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

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4. (a) Determine the maximum M of $x+y+z$ where x, y and z are positive real numbers with
$$16xyz = (x+y)^2(x+z)^2.$$
   (b) Prove the existence of infinitely many triples (x,y,z) of positive rational numbers that satisfy $16xyz = (x+y)^2(x+z)^2$ and $x+y+z = M$.

(Karl Czakler)

5. Let ABC be an acute triangle. Let H denote its orthocenter and D, E and F the feet of its altitudes from A, B and C, respectively. Let the common point of DF and the altitude through B be P. The line perpendicular to BC through P intersects AB in Q. Furthermore, EQ intersects the altitude through A in N.

   Prove that N is the mid-point of AH.

(Karl Czakler)

6. Let $S = \{1, 2, \ldots, 2017\}$.

   Find the maximal n with the property that there exist n distinct subsets of S such that for no two subsets their union equals S.

(Gerhard Woeginger)

Working time: $4\frac{1}{2}$ hours.
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