

48th Austrian Mathematical Olympiad
National Competition (Final Round, part 2, first day)
24th May 2017

1. Let α be a fixed real number.

Determine all functions $f: \mathbb{R} \rightarrow \mathbb{R}$ satisfying

$$f(f(x+y)f(x-y)) = x^2 + \alpha y f(y)$$

for all $x, y \in \mathbb{R}$.

(Walther Janous)

2. A necklace contains 2016 pearls, each of which has one of the colours black, green or blue. In each step we replace simultaneously each pearl with a new pearl, where the colour of the new pearl is determined as follows: If the two original neighbours were of the same colour, the new pearl has their colour. If the neighbours had two different colours, the new pearl has the third colour.

- (a) Is there such a necklace that can be transformed with such steps to a necklace of blue pearls if half of the pearls were black and half of the pearls were green at the start?
- (b) Is there such a necklace that can be transformed with such steps to a necklace of blue pearls if thousand of the pearls were black at the start and the rest green?
- (c) Is it possible to transform a necklace that contains exactly two adjacent black pearls and 2014 blue pearls to a necklace that contains one green pearl and 2015 blue pearls?

(Theresia Eisenkölbl)

3. Let $(a_n)_{n \geq 0}$ be the sequence of rational numbers with $a_0 = 2016$ and

$$a_{n+1} = a_n + \frac{2}{a_n}$$

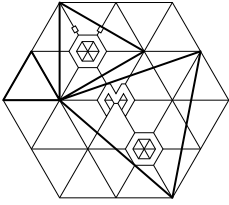
for all $n \geq 0$.

Show that the sequence does not contain a square of a rational number.

(Theresia Eisenkölbl)

Working time: $4\frac{1}{2}$ hours.

Each problem is worth 8 points.



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4. (a) Determine the maximum M of $x + y + z$ where x , y and z are positive real numbers with

$$16xyz = (x + y)^2(x + z)^2.$$

- (b) Prove the existence of infinitely many triples (x, y, z) of positive rational numbers that satisfy $16xyz = (x + y)^2(x + z)^2$ and $x + y + z = M$.

(Karl Czakler)

5. Let ABC be an acute triangle. Let H denote its orthocenter and D , E and F the feet of its altitudes from A , B and C , respectively. Let the common point of DF and the altitude through B be P . The line perpendicular to BC through P intersects AB in Q . Furthermore, EQ intersects the altitude through A in N .

Prove that N is the mid-point of AH .

(Karl Czakler)

6. Let $S = \{1, 2, \dots, 2017\}$.

Find the maximal n with the property that there exist n distinct subsets of S such that for no two subsets their union equals S .

(Gerhard Woeginger)

Working time: $4\frac{1}{2}$ hours.

Each problem is worth 8 points.