

**49<sup>th</sup> Austrian Mathematical Olympiad**  
National Competition (Final Round, part 2, first day)  
31st May 2018

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1. Let  $\alpha \neq 0$  be a real number.

Find all functions  $f: \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0}$  with

$$f(f(x) + y) = \alpha x + \frac{1}{f\left(\frac{1}{y}\right)}$$

for all  $x, y \in \mathbb{R}_{>0}$ .

*(Walther Janous)*

2. Let  $A, B, C$  and  $D$  be four different points lying on a common circle in this order. Assume that the line segment  $AB$  is the (only) longest side of the inscribed quadrilateral  $ABCD$ .

Prove that the inequality

$$AB + BD > AC + CD$$

holds.

*(Karl Czakler)*

3. There are  $n$  children in a room. Each child has at least one piece of candy. In Round 1, Round 2, etc., additional pieces of candy are distributed among the children according to the following rule:

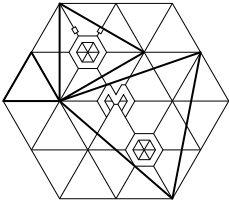
In Round  $k$ , each child whose number of pieces of candy is relatively prime to  $k$  receives an additional piece.

Show that after a sufficient number of rounds the children in the room have at most two different numbers of pieces of candy.

*(Theresia Eisenkölbl)*

Working time:  $4\frac{1}{2}$  hours.

Each problem is worth 8 points.



**49<sup>th</sup> Austrian Mathematical Olympiad**  
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1st June 2018

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4. Let  $ABC$  be a triangle and  $P$  a point inside the triangle such that the centers  $M_B$  and  $M_A$  of the circumcircles  $k_B$  and  $k_A$  of triangles  $ACP$  and  $BCP$ , respectively, lie outside the triangle  $ABC$ . In addition, we assume that the three points  $A$ ,  $P$  and  $M_A$  are collinear as well as the three points  $B$ ,  $P$  and  $M_B$ . The line through  $P$  parallel to side  $AB$  intersects circles  $k_A$  and  $k_B$  in points  $D$  and  $E$ , respectively, where  $D, E \neq P$ .

Show that  $DE = AC + BC$ .

*(Walther Janous)*

5. On a circle 2018 points are marked.

Each of these points is labeled with an integer. Let each number be larger than the sum of the preceding two numbers in clockwise order.

Determine the maximal number of positive integers that can occur in such a configuration of 2018 integers.

*(Walther Janous)*

6. Determine all digits  $z$  such that for each integer  $k \geq 1$  there exists an integer  $n \geq 1$  with the property that the decimal representation of  $n^9$  ends with at least  $k$  digits  $z$ .

*(Walther Janous)*

Working time:  $4\frac{1}{2}$  hours.

Each problem is worth 8 points.