50^{th} Austrian Mathematical Olympiad

National Competition—Final Round (Day 1)

29th May 2019

---

1. Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that

$$f(2x + f(y)) = x + y + f(x)$$

for all x, $y \in \mathbb{R}$.

(Gerhard Kirchner)

2. A (convex) trapezoid $ABCD$ shall be called *good* if it is inscribed, has parallel sides AB and CD, and CD is shorter than AB. For a good trapezoid, we fix the following notations.

   - The line parallel to AD through B intersects the line CD in S.
   - The tangents through S to the circumcircle of the trapezoid meet the circumcircle in E and F, respectively, where E is on the same side of the line CD as A.

   Characterize good trapezoids $ABCD$ (in terms of the side lengths and/or angles of the trapezoid) for which the angles $\angle BSE$ and $\angle FSC$ are equal. The characterization should be as simple as possible.

(Walther Janous)

3. In the country of Oddland, there are stamps with values 1 cent, 3 cent, 5 cent, etc., one type for each odd number. The rules of Oddland Postal Services stipulate the following: for any two distinct values, the number of stamps of the higher value on an envelope must never exceed the number of stamps of the lower value.

   In the country of Squareland, on the other hand, there are stamps with values 1 cent, 4 cent, 9 cent, etc., one type for each square number. Stamps can be combined in all possible ways in Squareland without additional rules.

   Prove for every positive integer n: In Oddland and Squareland there are equally many ways to correctly place stamps of a total value of n cent on an envelope. Rearranging the stamps on an envelope makes no difference.

(Stephan Wagner)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

50^{th} Austrian Mathematical Olympiad

National Competition—Final Round (Day 2)

30th May 2019

---

4. Let a, b and c be positive real numbers satisfying $a + b + c + 2 = abc$.

   Prove
   $$(a + 1)(b + 1)(c + 1) \geq 27.$$
   When does equality occur?

   (Karl Czakler)

5. We are given an arbitrary acute-angled triangle ABC and its altitudes AD and BE where D and E denote their feet on sides BC and AC, respectively. Let furthermore F and G be two points on segments AD and BE, respectively, such that
   $$\frac{AF}{FD} = \frac{BG}{GE}.$$
   The line through C and F intersects BE in point H and the line through C and G intersects AD in point I. Prove that the four points F, G, H and I are concyclic.

   (Walther Janous)

6. Determine the smallest possible positive integer n with the following property: For all positive integers x, y and z with $x \mid y^3$ and $y \mid z^3$ and $z \mid x^3$ we also have $xyz \mid (x+y+z)^n$.

   (Gerhard J. Woeginger)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.