52nd Austrian Mathematical Olympiad

National Competition—Final Round (Day 1)

4th June 2021

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1. Let a, b, c be pairwise distinct natural numbers.

   Prove that
   $$\frac{a^3+b^3+c^3}{3} \geq abc + a + b + c.$$

   When does equality hold?

   (Karl Czakler)

2. Mr. Precise wants to take his tea cup out of the microwave precisely at the front. The microwave of Mr. Precise is not precisely cooperative.

   More precisely, the two of them play the following game:

   Let n be a positive integer. The rotating plate of the microwave takes n seconds for a full turn. Each time the microwave is turned on, the plate is turned clockwise or counterclockwise for an integer number of seconds such that the tea cup can end up in n possible positions. One of these positions is marked „front".

   At the start of the game, the microwave rotates the tea cup in one of these positions. Afterwards, for each move, Mr. Precise enters the integer number of seconds and the microwave decides whether to turn clockwise or counterclockwise.

   For which n can Mr. Precise ensure that after a finite number of moves, he can take out the tea cup of the microwave precisely from the front position?

   (Birgit Vera Schmidt)

3. Determine all triples (a, b, c) of integers $a \geq 0$, $b \geq 0$ und $c \geq 0$ that satisfy the equation
   $$a^{b+20}(c-1) = c^{b+21} - 1.$$

   (Walther Janous)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.

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National Competition—Final Round (Day 2)

5th June 2021

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4. Let α be a real number.

   Determine all functions $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x) + y) = f(x^2 - y) + \alpha f(x)y$$

   for all $x, y \in \mathbb{R}$.

   (Walther Janous)

5. Let $ABCD$ be an inscribed convex quadrilateral with diagonals AC and BD. Each of the four vertices is reflected on the diagonal it does not lie on.

   Prove that the resulting four points lie on a common circle or a common line.

   (a) Investigate when the four resulting points lie on a common line and give a simple equivalent condition for the quadrilateral $ABCD$.

   (b) Prove that in all other cases, the four resulting points lie on a common circle.

   (Theresia Eisenkölbl)

6. Suppose that p is an odd prime number and M a set of $\frac{p^2+1}{2}$ integer squares.

   Investigate if one can choose p elements of this set so that the arithmetic mean of these p elements is an integer.

   (Walther Janous)

Working time: $4\frac{1}{2}$ hours.
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