56^{th} Austrian Mathematical Olympiad

Regional Competition

3rd April 2025

1. Let $n \geq 3$ be a positive integer. Furthermore, let $x_1, \ldots, x_n \in [0, 2]$ be real numbers subject to $x_1 + \ldots + x_n = 5$.

   Prove the inequality
   $$x_1^2 + \ldots + x_n^2 \leq 9.$$

   When does equality hold?

   (Walther Janous)

2. Let ABC be an isosceles triangle with $AC = BC$ and circumcircle k. The line through B perpendicular to BC is denoted by n. Furthermore, let M be any point on n. The circle k_1 with center M and radius BM intersects AB once more at point P and the circumcircle k once more at point Q.

   Prove that the points P, Q and C lie on a straight line.

   (Karl Czakler)

3. There are 6 different bus lines in a city, each stopping at exactly 5 stations and running in both directions. Nevertheless, for every two different stations there is always a bus line connecting these two stations. Determine the maximum number of stations in this city.

   (Karl Czakler)

4. Let z be a positive integer that is not divisible by 8. Furthermore, let $n \geq 2$ be a positive integer.

   Prove that none of the numbers of the form $z^n + z + 1$ is a square number.

   (Walther Janous)

Working time: 4 hours.
Each problem is worth 8 points.