48^{th} Austrian Mathematical Olympiad

Beginners' Competition

13th June 2017

1. The nonnegative real numbers a and b satisfy $a+b=1$. Prove that

$$\frac{1}{2} \leq \frac{a^3+b^3}{a^2+b^2} \leq 1.$$

When do we have equality in the right inequality and when in the left inequality?

(Walther Janous)

2. In the isosceles triangle ABC with $\overline{AC} = \overline{BC}$ we denote by D the foot of the altitude through C. The midpoint of CD is denoted by M. The line BM intersects AC in E. Prove that the length of AC is three times that of CE.

(Erich Windischbacher)

3. Anthony denotes in sequence all positive integers which are divisible by 2. Bertha denotes in sequence all positive integers which are divisible by 3. Claire denotes in sequence all positive integers which are divisible by 4. Orderly Dora denotes all numbers written by the other three. Thereby she puts them in order by size and does not repeat a number. What ist the 2017^{th} number in her list?

(Richard Henner)

4. How many solutions does the equation

$$\left\lfloor \frac{x}{20} \right\rfloor = \left\lfloor \frac{x}{17} \right\rfloor$$

have over the set of positve integers?

Therein $\lfloor a \rfloor$ denotes the largest integer that is less than or equal to a.

(Karl Czakler)

Working time: 4 hours.
Each problem is worth 8 points.