49^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 1)

28th April 2018

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1. Let α be an arbitrary positive real number. Determine for this number α the greatest real number C such that the inequality
$$\left(1+\frac{\alpha}{x^2}\right)\left(1+\frac{\alpha}{y^2}\right)\left(1+\frac{\alpha}{z^2}\right) \geq C \cdot \left(\frac{x}{z}+\frac{z}{x}+2\right)$$
is valid for all positive real numbers x, y and z satisfying $xy + yz + zx = \alpha$. When does equality occur?

2. Let ABC be a triangle with incenter I. The incircle of the triangle is tangent to the sides BC and AC in points D and E, respectively. Let P denote the common point of lines AI and DE, and let M and N denote the mid-points of sides BC and AB, respectively. Prove that points M, N and P are collinear.

3. Alice and Bob determine a number with 2018 digits in the decimal system by choosing digits from left to right. Alice starts and then they each choose a digit in turn. They have to observe the rule that each digit must differ from the previously chosen digit modulo 3.

   Since Bob will make the last move, he bets that he can make sure that the final number is divisible by 3. Can Alice avoid that?

4. Let M be a set containing positive integers with the following three properties:

   (1) $2018 \in M$.

   (2) If $m \in M$, then all positive divisors of m are also elements of M.

   (3) For all elements $k, m \in M$ with $1 < k < m$, the number $km + 1$ is also an element of M.

   Prove that $M = \mathbb{Z}_{\geq 1}$.

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.