50th Austrian Mathematical Olympiad

National Competition—Preliminary Round

4th May 2019

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1. We consider the sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$ which are defined by $a_0 = b_0 = 2$ and $a_1 = b_1 = 14$ and by

$$a_n = 14a_{n-1} + a_{n-2},$$
$$b_n = 6b_{n-1} - b_{n-2}$$

for $n \geq 2$.

Decide whether there are infinitely many integers which occur in both sequences.

(Gerhard Woeginger)

2. Let ABC be a triangle and I its incenter. The circumcircle of ACI intersects the line BC a second time in the point X and the circumcircle of BCI intersects the line AC a second time in the point Y.

Prove that the segments AY and BX are of equal length.

(Theresia Eisenkölbl)

3. Let $n \geq 2$ be an integer.

Ariane and Bérénice play a game on the set of residue classes modulo n. In the beginning, the residue class 1 is written on a piece of paper. In each move, the player whose turn it is replaces the current residue class x with either $x + 1$ or $2x$. The two players alternate with Ariane starting.

Ariane has won if the residue class 0 is reached during the game. Bérénice has won if she can permanently avoid this outcome.

For each value of n, determine which player has a winning strategy.

(Theresia Eisenkölbl)

4. Find all pairs (a, b) of real numbers such that

$$a \cdot \lfloor b \cdot n \rfloor = b \cdot \lfloor a \cdot n \rfloor$$

for all positive integers n.

(Walther Janous)

Working time: $4\frac{1}{2}$ hours.
Each problem is worth 8 points.