46^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 2) – Solutions

May 20, 2015 and May 21, 2015

Problem 1. *Let $f\colon \mathbb{Z}_{>0} \to \mathbb{Z}$ be a function with the following properties:*

(i) $f(1) = 0$,

(ii) $f(p) = 1$ *for all prime numbers* p,

(iii) $f(xy) = yf(x) + xf(y)$ *for all* x, y *in* $\mathbb{Z}_{>0}$.

Determine the smallest integer $n \geq 2015$ that satisfies $f(n) = n$.

(Gerhard J. Woeginger)

Solution.

1. We claim that
$$f(q_1 \cdots q_s) = q_1 \cdots q_s \left(\frac{1}{q_1} + \cdots + \frac{1}{q_s}\right) \tag{1}$$
holds for (not necessarily distinct) prime numbers $q_1, \ldots, q_s$.

   We prove the claim by induction on s. For $s = 0$, the claim reduces to $f(1) = 0$, which is true by assumption.

   If (1) holds for some s, then
$$\begin{aligned} f(q_1 \cdots q_s q_{s+1}) &= f((q_1 \cdots q_s) q_{s+1}) = q_{s+1} f(q_1 \cdots q_s) + q_1 \cdots q_s f(q_{s+1}) \\ &= q_1 \cdots q_{s+1}\Big(\frac{1}{q_1} + \cdots + \frac{1}{q_s}\Big) + q_1 \cdots q_s = q_1 \cdots q_{s+1}\Big(\frac{1}{q_1} + \cdots + \frac{1}{q_s} + \frac{1}{q_{s+1}}\Big). \end{aligned}$$

2. It is easily verified that the function given by (1) fulfills the given functional equation.

3. Let $p_1, \ldots, p_r$ be distinct primes and $\alpha_1, \ldots, \alpha_r$ be positive integers. Then collecting equal primes in (1) leads to
$$f(p_1^{\alpha_1} \cdots p_r^{\alpha_r}) = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \sum_{j=1}^{r} \frac{\alpha_j}{p_j}.$$

4. We now determine all $n \geq 2015$ with $f(n) = n$. We write $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$. Then
$$\frac{\alpha_1}{p_1} + \cdots + \frac{\alpha_r}{p_r} = 1. \tag{2}$$
We write
$$\frac{\alpha_1}{p_1} + \cdots + \frac{\alpha_{r-1}}{p_{r-1}} = \frac{a}{p_1 \cdots p_{r-1}}$$
for some non-negative integer a. Then
$$\frac{a}{p_1 \cdots p_{r-1}} + \frac{\alpha_r}{p_r} = 1 \iff a p_r + \alpha_r p_1 \cdots p_{r-1} = p_1 \cdots p_r.$$
As p_r is coprime to $p_1 \cdots p_{r-1}$, we conclude that $p_r \mid \alpha_r$. As (2) implies $\alpha_r \leq p_r$, we conclude that $r = 1$ and $\alpha_r = p_r$.

   Thus $f(n) = n$ holds if and only if $n = p^p$ for some prime number p. We have
$$2^2 = 4 < 3^3 = 27 < 2015 < 5^5 = 3125,$$
so the smallest such n is 3125.

(Clemens Heuberger) □

Problem 2. *We are given a triangle ABC. Let M be the mid-point of its side AB.*

Let P be an interior point of the triangle. We let Q denote the point symmetric to P with respect to M.

Furthermore, let D and E be the common points of AP and BP with sides BC and AC, respectively.

Prove that points A, B, D and E lie on a common circle if and only if $\angle ACP = \angle QCB$ holds.

(Karl Czakler)

Solution. Without loss of generality, we assume that P lies either on the segment CM or in the interior of the triangle AMC. If this is not the case, we exchange the names of vertices A and B and add the angle $\angle PCQ$ to both given angles.

Let C' be the point symmetric to C with respect to M. According to the assumptions of the problem, Q and B are the points symmetric to P and A with respect to M, respectively. Furthermore, let D' and E' be the common points of the lines $C'P$ and CP with the sides AC and AC', respectively.

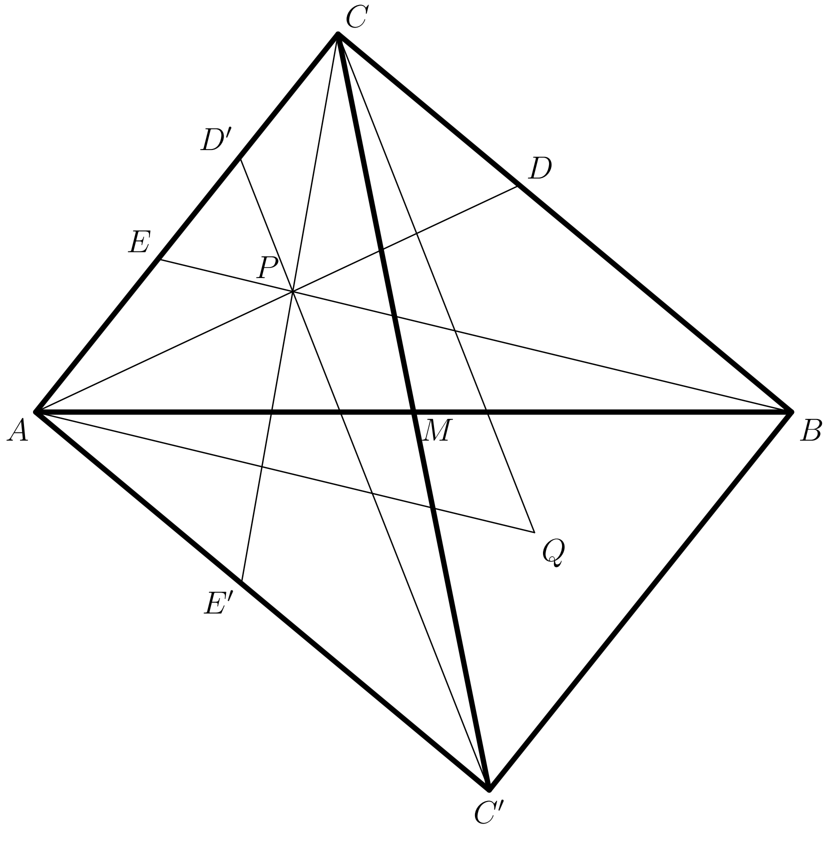

Figure 1: equivalence lemma

We first prove a lemma.

Lemma. Assume that P does not lie on the median CM. In this case, the following two facts hold:

1. Angles $\angle ACP$ and $\angle BCQ$ are equal if and only if the quadrilateral $C'CD'E'$ is circumscribed.
2. Angles $\angle CAP$ and $\angle C'AQ$ are equal if and only if the quadrilateral $ABDE$ is circumscribed.

Proof.

1. Due to symmetry with respect to M, we have $\angle BCQ = \angle AC'P$. Therefore

$$\begin{aligned} & \angle ACP = \angle BCQ \\ \iff & \angle ACP = \angle AC'P \\ \iff & \angle D'CE' = \angle D'C'E' \\ \iff & C'CD'E' \text{ is circumscribed} \end{aligned}$$

because of the equal angles on the chord $D'E'$.

2. Since $\angle C'AQ = \angle PBC$ also holds because of the symmetry, this follows as above.

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We first prove that if $ABDE$ is circumscribed, then $\angle ACP = \angle QCB$.

Let $ABDE$ be circumscribed, as in Figure 2. Since BQ lies symmetric to AP with respect to M,

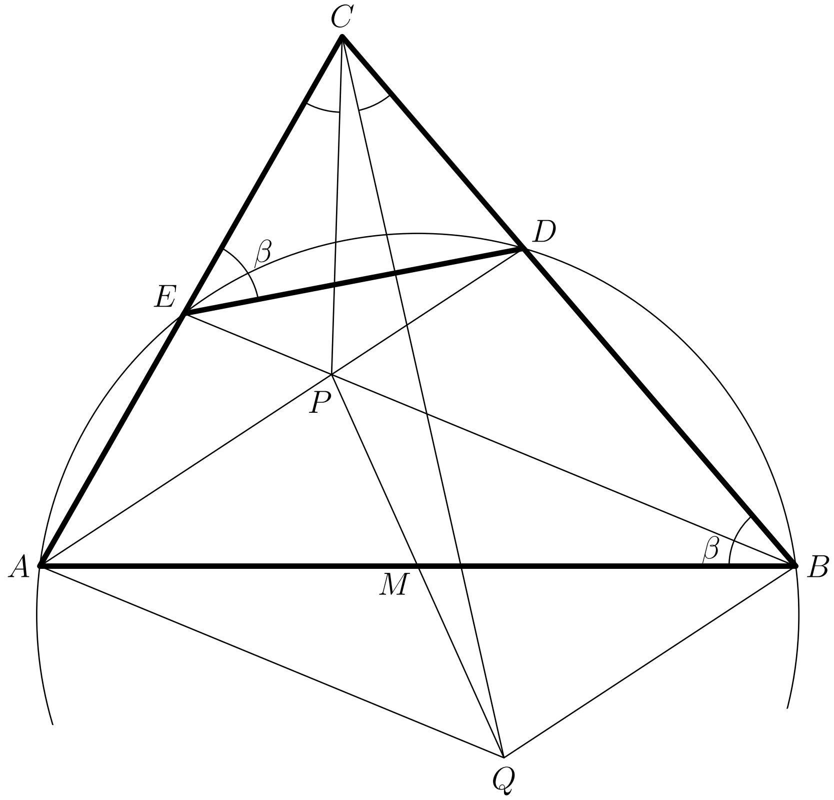

Figure 2: Problem 2

$AQBP$ is a parallelogram. Angle $\angle CED$ is supplementary to $\angle AED$, which is itself supplementary to $\angle ABC$ because the quadrilateral is circumscribed, and therefore $\angle CED = \angle ABC$ holds. It follows that triangles CED and CBA are similar.

We now reflect along the angle bisector of $\angle BCA$ and then perform a homothety, such that D is mapped onto A. Since the triangles CED and CBA are similar, this must map E onto B. Since AQB is congruent to BPA, which is itself similar to DPE, triangle DPE is mapped onto AQB, and therefore P onto Q. It therefore follows that $\angle ACP = \angle QCB$ holds, as required.

Now, we prove the converse direction under the additional assumption that P does not lie on CM. We assume that $\angle ACP = \angle BCQ$. The lemma then implies that $C'CD'E'$ is circumscribed. Applying the result on the first direction on the triangle $CC'A$ instead of ABC implies $\angle C'AQ = \angle CAP$. It therefore follows from the lemma that $ABDE$ lie on a common circle.

If P lies on CM and $\angle ACP = \angle BCQ$ holds, C, P, M and Q are all points on the angle bisector. It then follows that ABC is isosceles, and since P lies on the axis of symmetry, we see that $ED \parallel AB$ holds. It follows that $ABDE$ is an isosceles trapezoid, and it is therefore circumscribed.

(Levi Haunschmid, Sara Kropf) □

Problem 3. *We consider the following operation applied to a positive integer: The integer is represented in an arbitrary base $b \geq 2$, in which it has exactly two digits and in which both digits are different from 0. Then the two digits are swapped and the result in base b is the new number.*

Is it possible to transform every number > 10 to a number ≤ 10 with a series of such operations?

(Theresia Eisenkölbl)

Solution. We show that each number > 10 can be transformed to a smaller number. In that way, we will eventually reach a number ≤ 10.

If the number $n = 2k + 1$ is odd, we choose base $b = k$ with $n = (21)_k$. Swapping the two digits, we obtain the new number $(12)_k = k + 2$. Since $k \geq 5$, the choice of $b = k$ as base is admissible (the digits are smaller than the base) and we have $k + 2 \leq 2k - 5 + 2 < 2k + 1$ as desired.

If the number $n = 2k$ is even, we choose the base $b = 2k - 2$ with $n = (12)_{2k-2}$ and obtain the new number $(21)_{2k-2} = 4k - 3$. Now we choose the base $k - 1$ with $4k - 3 = (41)_{k-1}$ and obtain the new number $(14)_{k-1} = k + 3$. Since $k > 5$, both bases are admissible, and we have $k + 3 < 2k$ as desired.

(Theresia Eisenkölbl) □

Problem 4. *Let x, y, z be positive real numbers with $x + y + z \geq 3$. Prove that*

$$\frac{1}{x + y + z^2} + \frac{1}{y + z + x^2} + \frac{1}{z + x + y^2} \leq 1.$$

When does equality hold?

(Karl Czakler)

Solution. By Cauchy's inequality, we have

$$(x + y + z^2)(x + y + 1) \geq (x + y + z)^2, \tag{3}$$

hence

$$\frac{1}{x + y + z^2} \leq \frac{x + y + 1}{(x + y + z)^2}.$$

Thus it suffices to show that

$$\sum_{cyc} \frac{x + y + 1}{(x + y + z)^2} = \frac{2(x + y + z) + 3}{(x + y + z)^2} \leq 1.$$

This is equivalent to the inequality

$$(x + y + z)^2 - 2(x + y + z) - 3 \geq 0,$$

which holds for $x + y + z \geq 3$.

Equality in (3) holds if and only if (x, y, z^2) und $(x, y, 1)$ are collinear, i.e., $z^2 = 1$ or, equivalently, $z = 1$. Cyclic permutation shows that equality holds if and only if $x = y = z = 1$.

(Karl Czakler) □

Problem 5. *Let I be the incenter of triangle ABC and let k be a circle through the points A and B. This circle intersects*

- *the line AI in points A and P,*
- *the line BI in points B and Q,*
- *the line AC in points A and R and*
- *the line BC in points B and S,*

with none of the points A, B, P, Q, R und S coinciding and such that R and S are interior points of the line segments AC and BC, respectively.

Prove that the lines PS, QR and CI meet in a single point.

(Stephan Wagner)

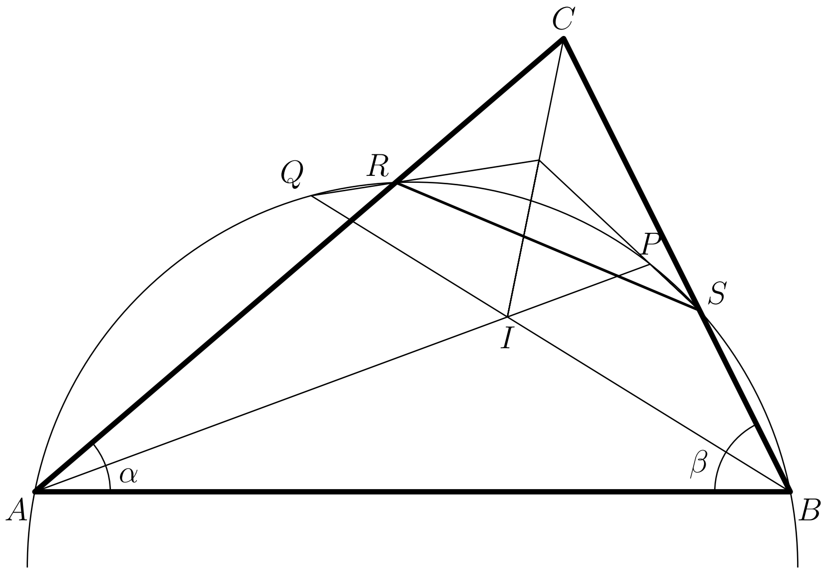

Figure 3: Problem 5

Solution. We define angles $\alpha = \angle BAC$ and $\beta = \angle CBA$ as usual, cf. Figure 3. Since points A, B, S and R lie on a common circle, we have $\angle BSR = 180° - \alpha$, and therefore $\angle RSC = \alpha$. Similarly, $\angle CRS = \beta$ also holds.

If P lies in the interior of ABC, we have $\angle RSP = \angle RAP = \alpha/2$. This means that PS bisects the angle $\angle CSR$.

If Q is outside of ABC, we have $\angle QRA = \angle QBA = \beta/2$, and in this case QR also bisects the angle $\angle SRC$.

Independent of the positioning of Q and R with respect to the triangle, we therefore see that QR, PS and CI are the bisectors of the interior angles of CRS, and they therefore meet in the incenter of this triangle, as claimed.

(Clemens Heuberger) □

Problem 6. *Max has* 2015 *jars labelled with the numbers* 1 *to* 2015 *and an unlimited supply of coins.*

Consider the following starting configurations:

(a) *All jars are empty.*

(b) *Jar 1 contains 1 coin, jar 2 contains 2 coins, and so on, up to jar 2015 which contains 2015 coins.*

(c) *Jar 1 contains 2015 coins, jar 2 contains 2014 coins, and so on, up to jar 2015 which contains 1 coin.*

Now Max selects in each step a number n from 1 to 2015 and adds n coins to each jar except to the jar n.

Determine for each starting configurations in (a), (b), (c), if Max can use a finite, strictly positive number of steps to obtain an equal number of coins in each jar.

(Birgit Vera Schmidt)

Solution. Max can achieve his goal in all three cases by the procedures described below.

Let $N = 2015$ be the number of jars.

(a) Let Max select jar j exactly $(N!/j)$ times. Then jar j will contain

$$\sum_{k \neq j} k \cdot \frac{N!}{k} = (N-1) \cdot N!$$

coins which does not depend on j as desired and has clearly needed at least one step.

(b) Let Max select each jar j exactly once. Then jar j will contain $j+\sum_{k\neq j} k = \sum_k k$ coins which does not depend on j as desired.

(c) Let Max select jar j exactly $(N!/j-1)$ times. Then jar j will contain

$$N+1-j+\sum_{k\neq j} k\cdot\left(\frac{N!}{k}-1\right) = N+1-j+\sum_{k\neq j}(N!-k) = (N-1)N!+(N+1)-\sum_{k} k$$

coins which does not depend on j as desired.

(Clemens Heuberger) □