48^{th} Austrian Mathematical Olympiad

National Competition (Final Round, part 2)—Solutions

24th/25th May 2017

Problem 1. *Let α be a fixed real number.*

Determine all functions $f\colon \mathbb{R} \to \mathbb{R}$ satisfying

$$f(f(x+y)f(x-y)) = x^2 + \alpha y f(y)$$

for all x, $y \in \mathbb{R}$.

(Walther Janous)

Solution. We will show that for $\alpha = -1$ the unique solution is $f(x) = x$ and for other values of α there is no solution.

Indeed, $x = y = 0$ yields $f(f(0)^2) = 0$. Furthermore, $x = 0$ and $y = f(0)^2$ imply $f(0) = 0$. Setting $y = x$, we get $f(0) = x^2 + \alpha x f(x)$. Now $\alpha = 0$ immediately leads to a contradiction, so from now on we assume $\alpha \neq 0$. Division by $x \neq 0$ results in $f(x) = -x/\alpha$ for $x \neq 0$. Because of $f(0) = 0$, this expression for $f(x)$ is valid for $x = 0$, too. Replacing f with this expression in the original equation gives $(x^2 - y^2)/(-\alpha)^3 = x^2 - y^2$ for all x, y which is equivalent to $-\alpha^3 = 1$, that is $\alpha = -1$, and the proof is complete.

(Theresia Eisenkölbl, Clemens Heuberger) □

Problem 2. *A necklace contains* 2016 *pearls, each of which has one of the colours black, green or blue. In each step we replace simultaneously each pearl with a new pearl, where the colour of the new pearl is determined as follows: If the two original neighbours were of the same colour, the new pearl has their colour. If the neighbours had two different colours, the new pearl has the third colour.*

(a) Is there such a necklace that can be transformed with such steps to a necklace of blue pearls if half of the pearls were black and half of the pearls were green at the start?

(b) Is there such a necklace that can be transformed with such steps to a necklace of blue pearls if thousand of the pearls were black at the start and the rest green?

(c) Is it possible to transform a necklace that contains exactly two adjacent black pearls and 2014 blue pearls to a necklace that contains one green pearl and 2015 blue pearls?

(Theresia Eisenkölbl)

Solution. (a) Since 2016 is divisible by 4, we can alternatingly take two black and two green pearls. In the first step, all pearls are already replaced by blue pearls.

(b) If we assign to each blue pearl the number 0, to each green pearl the number 1 and to each black pearl the number 2, then it holds in each step that the new colour of a pearl modulo 3 is equal to the negative sum of its two original neighbours. The new total sum of all colours modulo 3 therefore can be calculated by multiplying the old total sum of all colours with 2 and changing the sign. But modulo 3, a multiplication with -2 is equivalent to a multiplication with 1, therefore the total sum always remains the same modulo 3.

For a necklace with only blue pearls the total sum is 0. But for 1000 black and 1016 green pearls it is $2000 + 1016 \equiv 1 \pmod 3$. Therefore, there does not exist an arrangement of 1000 black and 1016 green pearls that can be transformed into a necklace with only blue pearls using such steps.

(c) Using the same assignment of numbers modulo 3, in each step the sum of all colours in even positions becomes the sum of the colours in odd positions, and vice versa. If these sums are A and B in the beginning, then at the end we still have these same two sums modulo 3, maybe with switched positions.

But in the beginning, we have sums 2 and 2 modulo 3, because both among the even and among the odd positions there is exactly one black pearl with value 2, and otherwise only blue pearls with value 0. However, at the end we are supposed to have sums 1 and 0 because one of the two sums is determined only by blue pearls with value 0, and the other by exactly one green pearl with value 1 and only blue pearls with value 0 otherwise. Therefore, it is not possible.

(Theresia Eisenkölbl) □

Problem 3. *Let $(a_n)_{n\geq 0}$ be the sequence of rational numbers with $a_0 = 2016$ and*

$$a_{n+1} = a_n + \frac{2}{a_n}$$

for all $n \geq 0$.

Show that the sequence does not contain a square of a rational number.

(Theresia Eisenkölbl)

Solution. We look at this sequence modulo 5. This is possible as long as $a_n \not\equiv 0 \pmod 5$ so that the next element is defined modulo 5. If we start to compute the elements modulo 5 we obtain

$$\begin{aligned}
a_0 &\equiv 1 \pmod 5,\\
a_1 &\equiv 1 + 2 \equiv 3 \pmod 5,\\
a_2 &\equiv 3 + 2 \cdot 3^{-1} \equiv 3 + 2 \cdot 2 \equiv 2 \pmod 5,\\
a_3 &\equiv 3 \pmod 5,\\
a_4 &\equiv 2 \pmod 5,\\
&\vdots
\end{aligned}$$

So we see that after a_0, the sequence just alternates between the values 2 and 3 modulo 5. These are not quadratic residues modulo 5. Since $a_0 = 2016$ is also not the square of a rational number, there is indeed no square of a rational number in this sequence.

(Theresia Eisenkölbl) □

Problem 4. *(a) Determine the maximum M of $x + y + z$ where x, y and z are positive real numbers with*

$$16xyz = (x+y)^2(x+z)^2.$$

(b) Prove the existence of infinitely many triples (x, y, z) of positive rational numbers that satisfy $16xyz = (x+y)^2(x+z)^2$ and $x + y + z = M$.

(Karl Czakler)

Solution. (a) The given equation and the AM-GM-inequality imply

$$4\sqrt{xyz} = (x+y)(x+z) = x(x+y+z) + yz \geq 2\sqrt{xyz(x+y+z)}.$$

Therefore, $2 \geq \sqrt{x+y+z}$ which gives $4 \geq x + y + z$. Since we will explicitly give infinitely many triples with $x + y + z = 4$ in the second part, $M = 4$ is the maximum.

(b) For $x+y+z=4$, equality must hold in the AM-GM-inequality of the first part, so we have $x(x+y+z)=yz$ and also $x+y+z=4$. If we choose $y=t$ with rational t we get $4x=t(4-x-t)$ and therefore $x=\frac{4t-t^2}{4+t}$ and $z=4-x-y=\frac{16-4t}{4+t}$. If we take $0<t<4$ then all these expressions are positive and rational and are a solution of the given equation.

The triples $\left(\frac{4t-t^2}{4+t}, t, \frac{16-4t}{4+t}\right)$ with rational $0<t<4$ are infinitely many cases of equality.

(Karl Czakler) □

Problem 5. *Let ABC be an acute triangle. Let H denote its orthocenter and D, E and F the feet of its altitudes from A, B and C, respectively. Let the common point of DF and the altitude through B be P. The line perpendicular to BC through P intersects AB in Q. Furthermore, EQ intersects the altitude through A in N.*

Prove that N is the mid-point of AH.

(Karl Czakler)

Solution. See Figure 1. As usual, let $\beta=\angle ABC$ and $\gamma=\angle ACB$. Since we know that $\angle AFH =$

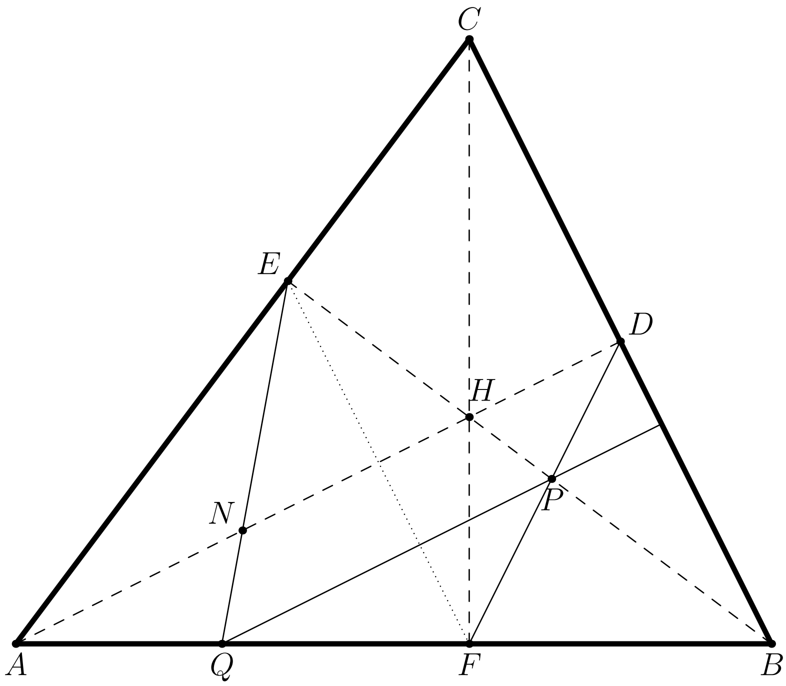

Figure 1: Problem 5

$\angle AEH = 90°$ holds, the quadrilateral $AFHE$ is cyclic, and because DA is parallel to PQ we obtain

$$\angle FQP = \angle FAH = \angle FEH = \angle FEP.$$

It follows that $QFPE$ is also cyclic. Since $\angle AFC = \angle ADC = 90°$, $AFDC$ is also cyclic, and we have $\angle QFP = \angle AFD = 180° - \angle ACD = 180° - \gamma$. We therefore have $\angle QEP = \gamma$. From this, we obtain $\angle EAN = 90° - \gamma = \angle AEP - \angle QEP = \angle AEN$, which shows us that triangle ANE is isosceles. It therefore follows that N is the circumcenter of the right triangle AHE, and we therefore have $NA = NH$, as claimed.

(Karl Czakler) □

Problem 6. *Let $S=\{1,2,\ldots,2017\}$.*

Find the maximal n with the property that there exist n distinct subsets of S such that for no two subsets their union equals S.

(Gerhard Woeginger)

Solution. Answer: $n = 2^{2016}$.

Proof:

There are 2^{2016} subsets of S which do not contain 2017. The union of any two such subsets does not contain 2017 and is thus a proper subset of S. Thus $n \geq 2^{2016}$.

To show the other direction, we group the subsets of S into 2^{2016} pairs in such a way that every subset forms a pair with its complement. If $n > 2^{2016}$ then the n subsets would contain such a pair. Its union would be S, contradiction.

Thus $n = 2^{2016}$.

(Gerhard Woeginger) □